

UNITARY CYCLES ON SHIMURA CURVES AND THE SHIMURA LIFT I: THE LOCAL SETTING

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ABSTRACT. This paper concerns two families of divisors, which we call the “orthogonal” and “unitary” special cycles, defined on arithmetic models associated to Shimura curves; in particular, we study their local structure at a prime of bad reduction. The orthogonal family was introduced by Kudla, Rapoport and Yang, who showed that their local components are the Fourier coefficients of a modular form of weight $3/2$. Our main result relates the Fourier coefficients of the Shimura lift of this form to the local components of unitary special cycles, which arise in recent work of Kudla and Rapoport on Shimura varieties of unitary type.

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1. INTRODUCTION

In a series of works leading up to the monograph [KRY], Kudla, Rapoport and Yang study a family of arithmetic divisors, which we refer to as “orthogonal cycles”, defined on integral models of Shimura curves. One of their main theorems states that if one assembles these divisors into a formal generating series, the result is *modular* of weight $3/2$; in effect, this means that pairing this generating series with a suitable linear functional yields the q -expansion of a weight $3/2$ modular form. In the present work, we consider the modular form obtained by intersecting the orthogonal cycles with an irreducible component of the Shimura curve at a prime p of bad reduction; our main result is an arithmetic interpretation of the *Shimura lift* of this form. More precisely, we show that the Fourier coefficients of the Shimura lift can be expressed in terms of arithmetic cycles associated to unitary groups. This result is in line with recent work (e.g. [KR3, How2, Ter2]) that exhibit

relationships between the latter “unitary” cycles and the Fourier coefficients of modular forms in a variety of contexts.

Let B be a rational indefinite quaternion algebra with discriminant D_B , and fix a maximal order \mathcal{O}_B . Consider the moduli stack \mathcal{C}_B over \mathbb{Z} which parametrizes pairs $\underline{A} = (A, \iota)$, consisting of an abelian surface A over some base scheme S , together with an \mathcal{O}_B -action $\iota : \mathcal{O}_B \rightarrow \text{End}(A)$ satisfying a determinant condition (cf. §3.1). This stack is then an integral model for the classical Shimura curve associated to B . For a positive integer n , Kudla, Rapoport and Yang define an “orthogonal” special cycle $\mathcal{Z}^o(n)$ as the moduli space of diagrams

$$\xi : A \rightarrow A,$$

where A is a \mathcal{O}_B -abelian surface (i.e. a point of \mathcal{C}_B), and ξ is a traceless \mathcal{O}_B -linear endomorphism of A such that $\xi^2 = -n$; we view $\mathcal{Z}^o(n)$ as a cycle on \mathcal{C}_B via the natural forgetful map. They then prove that for any irreducible component \mathcal{Y} of the fibre of \mathcal{C}_B at a prime p dividing D_B , the formal generating series

$$(1.1) \quad \Phi_{\mathcal{Y}}^o(\tau) := C + \sum_{n>0} \langle \mathcal{Z}^o(n), \mathcal{Y} \rangle q^n, \quad q = e^{2\pi i \tau},$$

is the q -expansion of a modular form of weight $3/2$, where $\langle \mathcal{Z}^o(n), \mathcal{Y} \rangle$ is an intersection multiplicity, and C is an explicit constant, see Theorem 3.7 below.

The “unitary” cycles, as constructed by Kudla and Rapoport [KR3], are also defined by a moduli problem. Let k be an imaginary quadratic field, with ring of integers \mathcal{O}_k , and suppose that there is an embedding $\phi : \mathcal{O}_k \rightarrow \mathcal{O}_B$. Fix a base scheme S , and suppose $\underline{E} = (E, i_E, \lambda_E)$ is a triple consisting of an elliptic curve E over S , an action $i_E : \mathcal{O}_k \rightarrow \text{End}(E)$, and a principal polarization λ_E compatible with the \mathcal{O}_k -action. Given a point $\underline{A} = (A, \iota) \in \mathcal{C}_B(S)$, we may consider the space of morphisms

$$\text{Hom}_{\mathcal{O}_k, \phi}(E, A) := \{y \in \text{Hom}_S(E, A) \mid y \circ i_E(a) = \iota(\phi(a)) \circ y \text{ for all } a \in \mathcal{O}_k\}.$$

As discussed in §3.1, this space admits a natural hermitian form $h_{E,A}^\phi$.

For an integer $m > 0$, we define the “unitary cycle” $\mathcal{Z}(m, \phi)$ as the moduli stack of tuples $(\underline{E}, \underline{A}, y)$, where \underline{E} and \underline{A} are as above and

$$(1.2) \quad y \in \text{Hom}_{\mathcal{O}_k, \phi}(E, A) \quad \text{such that } h_{E,A}^\phi(y, y) = m.$$

Again, we may view $\mathcal{Z}(m, \phi)$ as a cycle on \mathcal{C}_B via the natural forgetful map.

Our main theorem describes the relationship between these two families of cycles in terms of the *Shimura lift*, which is a classical operation on modular forms; it takes as its input a modular form F of half-integral weight together with a squarefree integer parameter t , and yields a modular form $Sh_t(F)$ of *even* integral weight. Moreover, when $F = \sum a(n)q^n$ is holomorphic, there is an explicit formula for the Fourier coefficients of $Sh_t(F)$ in terms of those Fourier coefficients of F that are of the form $a(tm^2)$.

Theorem 1.1 (see Theorem 3.8). *Suppose $k = \mathbb{Q}(\sqrt{\Delta})$, where $\Delta < 0$ is a squarefree even integer and assume further that every prime dividing D_B is inert in k . Let \mathcal{Y} be an irreducible component of the fibre of \mathcal{C}_B at a prime p dividing D_B , and define a generating series*

$$(1.3) \quad \Phi_{\mathcal{Y}}^u(\tau) := C' + \frac{1}{2h(k)} \sum_{m>0} \left(\sum_{[\phi] \in \mathcal{O}_k^\times / \mathcal{O}_B^\times} \left\langle \mathcal{Z}(m, \phi) + \mathcal{Z}\left(\frac{m}{\gcd(m, D_B)}, \phi\right), \mathcal{Y} \right\rangle \right) q^m,$$

where $q = e^{2\pi i\tau}$, the constant C' is an explicit constant, and the sum on $[\phi]$ is over any set of representatives for the \mathcal{O}_B^\times -equivalence classes of optimal embeddings, cf. the notation section of §3.

Then we have an equality of q -expansions:

$$(1.4) \quad Sh_{|\Delta|}(\Phi_{\mathcal{Y}}^o)(\tau) = \Phi_{\mathcal{Y}}^u(\tau),$$

where $Sh_{|\Delta|}$ is the Shimura lift with parameter $|\Delta|$. In particular, $\Phi_{\mathcal{Y}}^u$ is (the q -expansion of) a modular form of weight 2.

The key tool in the proof is the p -adic uniformization of the Shimura curve \mathcal{C}_B , which relates the formal completion along its fibre at p to (a formal model of) the Drinfeld p -adic upper half-plane \mathcal{D} . We also have p -adic uniformizations for the orthogonal and unitary special cycles, which are expressed in terms of linear combinations of analogous “local” cycles defined on \mathcal{D} . The proof of our main theorem then factors into a *local part*, where we consider the relationship between the local unitary and orthogonal cycles, and a *global part*, where we combine the p -adic uniformizations with the local results to show that the right hand side of (1.4) matches the aforementioned explicit formula for the Fourier expansion of the Shimura lift of $\Phi_{\mathcal{Y}}^o(\tau)$.

We now describe the contents of the paper in more detail. In Section 2, we consider the local aspect of the problem. Let $\mathbb{F} = \mathbb{F}_p^{alg}$ be an algebraic closure of \mathbb{F}_p , and let $W = W(\mathbb{F})$ the ring of Witt vectors. Denote by **Nilp** the category of W -schemes such that p is locally nilpotent, and for a scheme $S \in \mathbf{Nilp}$, let $\overline{S} := S \times_W \mathbb{F}$. Then the Drinfeld upper half-plane \mathcal{D} parametrizes tuples $\underline{X} = (X, \iota_X, \rho_X)$, where X is a p -divisible group of height 4 and dimension 2, over a base scheme $S \in \mathbf{Nilp}$, together with a “special” action $\iota_X : \mathcal{O}_{B,p} \rightarrow \text{End}(X)$ of the maximal order $\mathcal{O}_{B,p}$ (cf. §2.1); finally ρ_X is an $\mathcal{O}_{B,p}$ -linear quasi-isogeny

$$\rho_X : X \times_S \overline{S} \rightarrow \mathbb{X} \times_{\mathbb{F}} \overline{S}$$

of height 0, where \mathbb{X} is some fixed p -divisible group over \mathbb{F} endowed with a special $\mathcal{O}_{B,p}$ -action. In this picture, the local analogues of the orthogonal and unitary cycles are described as the *deformation loci* of homomorphisms of p -divisible groups. A complete description of the orthogonal cycles can be found in [KR1], and so our aim in Section 2 is provide the same for the unitary cycles.

Recent work of Kudla and Rapoport [KR4] gives a description of the special fibre of \mathcal{D} in terms of the Bruhat-Tits tree for $SU(C)$, where C is the split 2-dimensional hermitian space over k_p (recall that by assumption, k_p is an unramified quadratic extension of \mathbb{Q}_p). Combining this description with a healthy dose of Grothendieck-Messing theory, we are able to write down explicit equations for a local unitary cycle, and we consequently obtain a precise description of its irreducible components. This description parallels the one found in [KR1] for the orthogonal cycles, and by comparing the two formulas, we obtain the following key result (Theorems 2.17 and 2.19): any local orthogonal cycle that appears in the p -adic uniformization of an orthogonal cycle $\mathcal{Z}^o(|\Delta|n^2)$ can be expressed as sum of two (explicitly determined) unitary cycles.

In Section 3, we discuss the global situation. After briefly introducing the global moduli problems and the Shimura lift in the first two subsections, we describe the p -adic uniformizations of the Shimura curve and the special cycles. These can be summarized as follows. If we fix an \mathbb{F} -valued point $\underline{\mathbf{A}} = (\mathbf{A}, \iota_{\mathbf{A}}) \in \mathcal{C}_B(\mathbb{F})$, then the space of \mathcal{O}_B -linear quasi-endomorphisms

$$B' := \text{End}_{\mathcal{O}_B}(\mathbf{A})_{\mathbb{Q}}$$

is a definite quaternion algebra over \mathbb{Q} with discriminant D_B/p . The p -adic uniformization of the Shimura curve \mathcal{C}_B can be expressed in the following way: there is a finite subgroup $\Gamma' \subset (B')^\times$

acting on \mathcal{D} , such that if we let $\widetilde{\mathcal{C}}_{\mathbf{B}}$ denote the base change to W of the formal completion of $\mathcal{C}_{\mathbf{B}}$ along its fibre at p , then there is an isomorphism

$$\widetilde{\mathcal{C}}_{\mathbf{B}} \simeq [\Gamma' \backslash \mathcal{D}]$$

of formal stacks over W .

Similarly, we let $\widetilde{\mathcal{Z}}^o(n)$ and $\widetilde{\mathcal{Z}}(m, \phi)$ denote the base change to W of the formal completions of the special cycles $\mathcal{Z}^o(n)$ and $\mathcal{Z}(m, \phi)$ along their fibres at p . Viewed as a cycle on $\widetilde{\mathcal{C}}_{\mathbf{B}}$, we may express an orthogonal cycle as a sum

$$\widetilde{\mathcal{Z}}^o(n) = \sum_{\substack{\xi \in \Omega^o(n) \\ \text{mod } \Gamma'}} [Z^o(\xi[p^\infty])]$$

where $[Z^o(\xi[p^\infty])]$ is the projection to $[\Gamma' \backslash \mathcal{D}]$ of a *local orthogonal cycle* on \mathcal{D} , and

$$\Omega^o(n) \subset \{b' \in B' \mid \text{Tr}(b') = 0\}$$

is a Γ' -invariant set of vectors of reduced norm n satisfying a certain integrality property, cf. Theorem 3.11. For the unitary cycles, we fix a triple $\underline{\mathbf{E}} = (\mathbf{E}, i_{\mathbf{E}}, \lambda_{\mathbf{E}})$ consisting of a supersingular elliptic curve \mathbf{E} over \mathbb{F} with an action $i_{\mathbf{E}} : o_k \rightarrow \text{End}(\mathbf{E})$, and a compatible principal polarization $\lambda_{\mathbf{E}}$. A unitary cycle then decomposes as

$$\widetilde{\mathcal{Z}}(m, \phi) = \frac{1}{|o_k^\times|} \sum_{[\mathfrak{a}] \in Cl(k)} \left(\sum_{\substack{\beta \in \Omega^+(m, \mathfrak{a}, \phi) \\ \text{mod } \Gamma'}} [Z(\beta[p^\infty])] + \sum_{\substack{\beta' \in \Omega^-(m, \mathfrak{a}, \phi) \\ \text{mod } \Gamma'}} [Z(\beta'[p^\infty])] \right),$$

where $Cl(k)$ is the class group of k , and as before $[Z(\beta[p^\infty])]$ and $[Z(\beta'[p^\infty])]$ are local unitary cycles. The sets $\Omega^\pm(m, \mathfrak{a}, \phi)$ appearing above are subsets

$$\Omega^\pm(m, \mathfrak{a}, \phi) \subset \text{Hom}(\mathbf{E}, \mathbf{A}) \otimes_{\mathbb{Z}} \mathbb{Q},$$

consisting of quasi-morphisms of a specified norm, a linearity (or anti-linearity, in the case of Ω^-) condition with respect to the action of o_k , and again satisfying an integrality condition, cf. Theorem 3.12.

Thus, in order to compare the unitary and orthogonal cycles, we need to compare the indexing sets $\Omega^o(n)$ and $\Omega^\pm(m, \mathfrak{a}, \phi)$, at least in the case that the squarefree part of n is equal to t , and as ϕ varies among classes of optimal embeddings. This task, which amounts to an étude in the arithmetic of quaternion algebras, is carried out in §3.4. Together with the description of the local cycles discussed previously, we arrive at a relationship between the two families of cycles that matches exactly the one implied by the formula for the Fourier coefficients of the Shimura lift, proving our main theorem.

I would like to conclude the introduction with an advertisement for the companion [San2] to the present work. As mentioned before, the modularity of the generating series (1.1) is a small part of a much larger theorem [KRY, Theorem A] which asserts that the generating series (for $\tau \in \mathbb{C}$ with $v = \text{Im}(\tau) > 0$)

$$\widehat{\Phi}^o(\tau) := \sum_{n \in \mathbb{Z}} \widehat{\mathcal{Z}}^o(n, v) q^n, \quad q = e^{2\pi i \tau}$$

is a non-holomorphic modular form of weight $3/2$. Here the coefficients are *arithmetic classes*

$$\widehat{\mathcal{Z}}^o(n, v) = (\mathcal{Z}^o(n), \text{Gr}^o(n, v)) \in \widehat{CH}^1(\mathcal{C}_{\mathbf{B}})_{\mathbb{R}}$$

in the first arithmetic Chow group of \mathcal{C}_B (with real coefficients), for appropriate choices of Green functions $Gr^o(n, v)$, and constant term $\widehat{\mathcal{Z}}^o(0, v)$.

Explicit Green functions $Gr(m, v, \phi)$ for the unitary cycles were constructed in [San], and with these in hand, we obtain classes

$$\widehat{\mathcal{Z}}(m, v, \phi) = (\mathcal{Z}(m, v, \phi), Gr(m, v, \phi)) \in \widehat{CH}^1(\mathcal{C}_B)_{\mathbb{R}}$$

which, together with an appropriate constant term $\mathcal{Z}(0, v, \phi)$, we then string together to form the generating series

$$\widehat{\Phi}^u(\tau) := \frac{1}{2h(k)} \sum_{[\phi] \in \mathcal{O}_{pt}/\mathcal{O}_B^\times} \sum_{m \in \mathbb{Z}} \left[\widehat{\mathcal{Z}}(m, v, \phi) + \widehat{\mathcal{Z}}\left(\frac{m}{\gcd(m, D_B)}, v, \phi\right) \right] q^m.$$

Combining the results of the present paper with calculations in the Archimedean components yields the main result of [San2]: under the same assumptions as Theorem 1.1, the generating series $\widehat{\Phi}^u(\tau)$ is the q -expansion of the Shimura lift $Sh_{|\Delta|}(\widehat{\Phi}^o)(\tau)$. Such a statement can be viewed as strong evidence for the modularity of a conjectural generating series of arithmetic cycles on a Shimura variety of type $GU(1, 1)$.

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2. UNITARY CYCLES ON THE DRINFELD UPPER HALF-PLANE

Notation: Let $p \neq 2$ be a prime number, and k_p the quadratic unramified extension of \mathbb{Q}_p . Denote by $a \mapsto a'$ the non-trivial Galois automorphism of k_p , and write $k_p = \mathbb{Q}_p(\delta)$, where $\delta \in k_p^\times$ with $\delta' = -\delta$.

Let B_p denote the division quaternion algebra over \mathbb{Q}_p , and fix an embedding

$$(2.1) \quad \phi : \mathcal{O}_{k,p} \hookrightarrow \mathcal{O}_{B,p},$$

where $\mathcal{O}_{k,p}$ and $\mathcal{O}_{B,p}$ are the maximal orders in k_p and B_p respectively. We also fix a uniformizer $\Pi \in \mathcal{O}_{B,p}$ such that $\Pi\phi(a) = \phi(a')\Pi$, for all $a \in \mathcal{O}_{k,p}$.

Let $\mathbb{F} = \mathbb{F}_p^{alg}$ denote a fixed algebraic closure of \mathbb{F}_p , and $W = W(\mathbb{F})$ the ring of Witt vectors. We fix an embedding $\tau_0 : \mathcal{O}_{k,p}/(p) \rightarrow \mathbb{F}$, which lifts uniquely to an embedding $\tau_0 : \mathcal{O}_{k,p} \rightarrow W$. In addition, we let τ_1 denote the conjugate embedding (or its lift to characteristic 0).

Finally, let **Nilp** denote the category of W -schemes for which the ideal sheaf generated by p is locally nilpotent, and for a scheme $S \in \mathbf{Nilp}$, we set

$$\overline{S} := S \times_W \text{Spec}(\mathbb{F}).$$

2.1. Drinfeld space. In this section, we recall the definition of Drinfeld space as a moduli space of p -divisible groups, as well as an “alternative” description of its special fibre as a union of projective lines indexed by *hermitian* lattices, as in [KR4].

Following [BC], we define a *special formal $\mathcal{O}_{B,p}$ -module* over a scheme S to be a pair (X, ι_X) consisting of a p -divisible group X of height 4 and dimension 2 over S , and a map

$$\iota_X : \mathcal{O}_{B,p} \rightarrow \text{End}(X),$$

which satisfies the following *special condition*: the Lie algebra $Lie(X)$ is (locally on S) a free $\mathcal{O}_S \otimes_{W, \tau_0} \mathcal{O}_{k,p}$ module of rank 1. Here $Lie(X)$ is viewed as an $\mathcal{O}_S \otimes \mathcal{O}_{k,p}$ -module via the action of $\mathcal{O}_{k,p}$ on X induced by the composition $\iota_X \circ \phi$.

Fix once and for all a special formal $\mathcal{O}_{B,p}$ -module $(\mathbb{X}, \iota_{\mathbb{X}})$ over \mathbb{F} ; note such a pair is unique up to isogeny, by the classification of Dieudonné isocrystals over \mathbb{F} [BC, Proposition II.5.2]. The pair $(\mathbb{X}, \iota_{\mathbb{X}})$ serves as a “base point” for the following moduli problem:

Definition 2.1. *Let \mathcal{D} denote the moduli problem on \mathbf{Nilp} , which associates to a scheme S the category of isomorphism classes of tuples*

$$\mathcal{D}(S) = \{(X, \iota_X, \rho_X)\} / \simeq,$$

consisting of

- (X, ι_X) a special formal \mathcal{O}_B -module over S ,
- and an $\mathcal{O}_{B,p}$ -linear quasi-isogeny

$$\rho_X : X \times_S \overline{S} \rightarrow \mathbb{X} \times_{\mathrm{Spec}(\mathbb{F})} \overline{S},$$

of height 0.

An isomorphism between two tuples (X, ι, ρ) and (X', ι', ρ') is an isomorphism $\alpha : X \rightarrow X'$ which is $\mathcal{O}_{B,p}$ -equivariant, and such that $\rho = \rho' \circ (\alpha \times_S S_0)$.

The functor \mathcal{D} is then represented by the (formal model of) the Drinfeld upper-half plane, and in particular is a formal scheme over $\mathrm{Spf} W$; see [BC] for a discussion of this result.

Next, we recall the ‘alternative’ description of the reduced locus of \mathcal{D} as given in [KR4]; crucial to this description is the following theorem:

Theorem 2.2 (Drinfeld, cf. §III.4 of [BC]). *Suppose (X, ι_X) is a special formal $\mathcal{O}_{B,p}$ -module, over any base $S \in \mathbf{Nilp}$. Then there exists a principal polarization λ_X^0 on X such that*

$$(2.2) \quad (\lambda_X^0)^{-1} \circ \iota_X(b)^\vee \circ \lambda_X^0 = \iota_X(\Pi b^\vee \Pi^{-1}) \quad \text{for all } b \in \mathcal{O}_{B,p};$$

here the map $b \mapsto b^\vee$ is the involution of B . Moreover, λ_X^0 is unique up to multiplication by \mathbb{Z}_p^\times .

For the base point \mathbb{X} , we shall fix once and for all a polarization $\lambda_{\mathbb{X}}^0$ as in Drinfeld’s theorem. Then, for any point $(X, \iota_X, \rho_X) \in \mathcal{D}(S)$, there is a *unique* principal polarization λ_X^0 satisfying (2.2), and such that the diagram

$$\begin{array}{ccc} X \times_S \overline{S} & \xrightarrow{\lambda_X^0 \times \overline{S}} & X^\vee \times_S \overline{S} \\ \rho_X \downarrow & & \downarrow \rho_X^\vee \\ \mathbb{X} \times_{\mathbb{F}} \overline{S} & \xrightarrow{\lambda_{\mathbb{X}}^0 \times \overline{S}} & \mathbb{X}^\vee \times_{\mathbb{F}} \overline{S} \end{array}$$

commutes. Thus, for any point $(X, \iota_X, \rho) \in \mathcal{D}(S)$, we may define another polarization $\lambda_{X,\phi}$ by the formula

$$(2.3) \quad \lambda_{X,\phi} := \lambda_X^0 \circ \iota_X(\Pi \phi(\delta)).$$

Note that the Rosati involution $*$ induced by $\lambda_{X,\phi}$ has the property

$$\iota_X(\phi(a))^* = \iota_X(\phi(a')), \quad \text{for all } a \in \mathcal{O}_{k,p}.$$

Let $M(\mathbb{X})$ be the Dieudonné module of \mathbb{X} over $W = W(\mathbb{F})$. Then we have a decomposition

$$M(\mathbb{X}) = M(\mathbb{X})_0 \oplus M(\mathbb{X})_1,$$

where

$$(2.4) \quad M(\mathbb{X})_i := \{m \in M(\mathbb{X}) \mid (\iota_{\mathbb{X}} \circ \phi)(a) \cdot m = \tau_i(a)m, \text{ for all } a \in o_{k,p}\}.$$

Let $N(\mathbb{X}) := M(\mathbb{X}) \otimes_{\mathbb{Z}} \mathbb{Q}$ be the rational Dieudonné module, with induced decomposition

$$N(\mathbb{X}) = N(\mathbb{X})_0 \oplus N(\mathbb{X})_1.$$

Note that the operator $pV^{-2} = V^{-1}F : N(\mathbb{X})_0 \rightarrow N(\mathbb{X})_0$ is a σ^2 -linear operator, and hence the space of invariants

$$(2.5) \quad C := (N(\mathbb{X})_0)^{V^{-1}F=1}$$

is a two-dimensional vector space over k_p . Here k_p acts on C via the embedding $\tau_0 : k_p \rightarrow W$.

The polarization $\lambda_{\mathbb{X},\phi}$, as defined in (2.3), induces an alternating pairing

$$\{\cdot, \cdot\}_{\mathbb{X}} : N(\mathbb{X}) \times N(\mathbb{X}) \rightarrow W_{\mathbb{Q}}$$

such that for all $x, y \in N(\mathbb{X})$, we have

$$\{Fx, y\}_{\mathbb{X}} = \sigma(\{x, Vy\}_{\mathbb{X}}).$$

Thus, if we define

$$(2.6) \quad h(x, y) := \frac{1}{p\delta} \{x, Fy\}_{\mathbb{X}},$$

it is straightforward to verify that the restriction of h to C defines a hermitian form; we denote this restricted form again by h . It is also easy to check that (C, h) is in fact *split*. Let $q(x) = h(x, x)$ denote the corresponding quadratic form.

Definition 2.3. (i) If L is a W -lattice in $N(\mathbb{X})_0$, we let

$$L^{\sharp} := \{n \in N(\mathbb{X})_0 \mid h(n, L) \subset W\}.$$

Note that $(L^{\sharp})^{\sharp} = pV^{-2}L$. Similarly, if $\Lambda \subset C$ is an $o_{k,p}$ -lattice, we set

$$\Lambda^{\sharp} := \{v \in C \mid h(v, \Lambda) \subset o_{k,p}\}.$$

(ii) Suppose $\Lambda \subset C$. We say Λ is a vertex lattice of type 0 (resp. type 2) if $\Lambda^{\sharp} = \Lambda$ (resp. $\Lambda^{\sharp} = p\Lambda$). We shall use the term “vertex lattice” to mean a vertex lattice of type 0 or 2.

(iii) Let \mathcal{B} denote the Bruhat-Tits tree for $SU(C)$, which is a graph with the following description. The vertices are vertex lattices, and edges can only occur between vertex lattices of differing type. Two vertex lattices Λ and Λ' of type 0 and 2 respectively are joined by an edge if and only if

$$p\Lambda' \subset \Lambda \subset \Lambda',$$

where the successive quotients are \mathbb{F}_{p^2} vector spaces of dimension 1. In particular, this graph is a $p+1$ -regular tree.

Suppose $x = (X, \iota_X, \rho_X) \in \mathcal{D}(\mathbb{F})$. We may use the quasi-isogeny ρ_X to identify the Dieudonné module $M(X)$ as a W -lattice inside of $N(\mathbb{X})$. Furthermore, as ρ_X is $o_{k,p}$ -linear, we have $M(X)_i = M(X) \cap N(\mathbb{X})_i$ for $i = 0, 1$, where $M(X)_i$ is defined in the same way as (2.4). Hence, to any point x , we may associate a chain of W -lattices $B \subset A$, where

$$B = M(X)_0, \quad A = (VM(X)_1)^{\sharp}.$$

By [KR4, Corollary 2.3], we have either $B^{\sharp} = B$ or $A^{\sharp} = pA$, or both. If both conditions are satisfied, then we say the point x is *superspecial*; otherwise, x is *ordinary*. We say a point is *special* if both A and B are pV^{-2} -invariant, so in particular superspecial points are special.

This construction yields a bijection between $\mathcal{D}(\mathbb{F})$ and pairs of W -lattices $B \subset A$ such that either $B^\sharp = B$ or $A^\sharp = pA$. Moreover, if $B^\sharp = B$, then $B = \Lambda \otimes_{\mathcal{O}_{k,p}} W$ for some vertex lattice Λ of type 0; on the other hand, if $A^\sharp = pA$, then $A = \Lambda' \otimes W$ for a vertex lattice Λ' of type 2, c.f. [KR4, Corollary 2.3].

Suppose Λ is a vertex lattice of type 0. We may define a map

$$\mathbb{P}_\Lambda(\mathbb{F}) := \mathbb{P}(p^{-1}\Lambda/\Lambda)(\mathbb{F}) \rightarrow \mathcal{D}(\mathbb{F}),$$

by sending a line $\ell \subset (p^{-1}\Lambda/\Lambda) \otimes_{\mathbb{F}_{p^2}} \mathbb{F}$ to the pair of lattices $B \subset A$, where $B = \Lambda_W = \Lambda \otimes W$, and A is the inverse image of ℓ in $p^{-1}\Lambda_W$.

If Λ' is a vertex lattice of type 2, we obtain a map

$$\mathbb{P}_{\Lambda'}(\mathbb{F}) := \mathbb{P}(\Lambda'/p\Lambda')(\mathbb{F}) \rightarrow \mathcal{D}(\mathbb{F}),$$

defined by sending a line $\ell' \subset (\Lambda'/p\Lambda') \otimes_{\mathbb{F}_{p^2}} \mathbb{F}$ to the pair of lattices $B \subset A$, where $A = \Lambda'_W$, and B is the inverse image of ℓ' in Λ'_W .

Note that if Λ and Λ' are neighbours in \mathcal{B} , i.e. if $p\Lambda' \subset \Lambda \subset \Lambda'$, then the lines

$$\ell = \Lambda' \otimes_{\mathcal{O}_{k,p}, \tau_0} \mathbb{F} \in \mathbb{P}(p^{-1}\Lambda/\Lambda)(\mathbb{F}), \quad \ell' = \Lambda \otimes \mathbb{F} \in \mathbb{P}(\Lambda'/p\Lambda')(\mathbb{F})$$

define the same point of $\mathcal{D}(\mathbb{F})$; this point is superspecial, and all superspecial points arise in this way.

By [KR4, Proposition 2.4], the above maps are induced by embeddings of schemes over \mathbb{F} :

$$\mathbb{P}_\Lambda \rightarrow \mathcal{D}_{red}, \quad \Lambda \text{ a vertex lattice,}$$

where \mathcal{D}_{red} is the underlying reduced subscheme of the formal scheme \mathcal{D} , and the collection of such maps, as Λ varies among the vertex lattices, yield a cover of \mathcal{D}_{red} by projective lines.

Remark 2.4. (i) In [KR1, §1], there is a similar description of the special fibre of \mathcal{D} , but it is given in terms of homothety classes of \mathbb{Z}_p -lattices. These two descriptions are essentially the same - in particular, we presently exhibit a natural bijection between the set of homothety classes of \mathbb{Z}_p lattices in \mathbb{Q}_p^2 , and vertex lattices in C . To start, note that the operator

$$\epsilon := \Pi_{\mathbb{X}}^{-1} \circ V : N(\mathbb{X}) \rightarrow N(\mathbb{X}), \quad \Pi_{\mathbb{X}} := \iota_{\mathbb{X}}(\Pi)$$

is 0-graded and commutes with $F^{-1}V$, and hence restricts to a Galois-semilinear operator $\epsilon : C \rightarrow C$.

Without loss of generality, we may assume $\mathbb{X} = \mathbb{Y} \times \mathbb{Y}$, where \mathbb{Y} is a supersingular p -divisible group of height 2 and dimension 1 over \mathbb{F} . Then the Dieudonné module $M(\mathbb{X})$ has a basis $\{e_0, e_1, f_0, f_1\}$ consisting of vectors that are both $F^{-1}V$ - and ϵ -invariant, and such that

$$M(\mathbb{X})_0 = W \cdot e_0 \oplus W \cdot f_1, \quad M(\mathbb{X})_1 = W \cdot e_1 \oplus W \cdot f_0,$$

c.f. [BC, §III.4.5]. In particular, the set $\{e_0, f_1\}$ is an ϵ -invariant k_p -basis for C such that $h(e_0, f_1) = \delta$ and $h(e_0, e_0) = h(f_1, f_1) = 0$. Let $\Lambda_0 := \text{span}_{\mathcal{O}_{k,p}}(e_0, f_1)$, which is a vertex lattice of type 0, i.e. $\Lambda_0^\sharp = \Lambda_0$.

Now suppose $\gamma \in \text{End}_{\mathcal{O}_{B,p}}(\mathbb{X}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Then γ also acts on C , and is described by a matrix of the form

$$[\gamma] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{Q}_p.$$

with respect to the basis $\{e_0, f_1\}$. In particular (at least, when γ is invertible), the matrix $[\gamma]$ lies in $GU(C)$ with $[\gamma]^* = \det(\gamma) \cdot [\gamma]^{-1}$. One can then verify directly that the map $\gamma \mapsto [\gamma]$ induces an

isomorphism

$$(2.7) \quad \{\gamma \in \text{End}_{\mathcal{O}_{B,p}}(\mathbb{X})_{\mathbb{Q}_p}, \det(\gamma) = 1\} \xrightarrow{\sim} SU(C).$$

Suppose $[L]$ is a homothety class of \mathbb{Z}_p -lattices in $M(\mathbb{X})_0^{\epsilon=1} = C^{\epsilon=1} \simeq (\mathbb{Q}_p)^2$, and $L \in [L]$. Then there exists $\gamma \in \text{End}_{\mathcal{O}_{B,p}}(\mathbb{X})_{\mathbb{Q}_p}$ such that $\gamma(L) = L_0 := \text{span}_{\mathbb{Z}_p}\{e_0, f_1\}$. Set $\Lambda := L \otimes_{\mathbb{Z}_p} o_{k,p}$, so that

$$\Lambda^\sharp = ([\gamma]^*)^{-1} \cdot \Lambda_0^\sharp = \det(\gamma)^{-1} [\gamma] \cdot \Lambda_0 = \det(\gamma)^{-1} \Lambda.$$

Hence, by scaling by a power of p , there is a unique representative $L \in [L]$ such that $L \otimes o_{k,p}$ is a vertex lattice (whose type depends on the parity of $\text{ord}_p \det(\gamma)$).

Conversely, suppose $\Lambda \subset C$ is a vertex lattice. Since $SU(C)$ acts transitively on the set of lattices of a given type, there exists an element $[\gamma] \in SU(C)$ such that $[\gamma] \cdot \Lambda$ is equal to one of Λ_0 or $\Lambda'_0 := \text{span}_{o_{k,p}}(p^{-1}e_0, f_1)$, depending on the type of Λ . In either case, by (2.7), the transformation $[\gamma]$ commutes with ϵ , and so Λ admits an ϵ -invariant basis. Thus the \mathbb{Z}_p -lattice $\Lambda^{\epsilon=1}$ determines a homothety class of \mathbb{Z}_p -lattices $[L] := [\Lambda^{\epsilon=1}]$.

(ii) Suppose $x = (X, \iota_X, \rho_X) \in \mathcal{D}(\mathbb{F})$, and let $M = M_0 \oplus M_1$ denote its Diedonné module, where each M_i is viewed as a W -lattice in $N(\mathbb{X})_i$. Each lattice M_i is self-dual with respect to the pairing induced by the polarization λ_X^0 described in Theorem 2.2, and it is then a straightforward calculation to show

$$M_0^\sharp = \Pi^{-1} V M_0, \quad \text{and} \quad (V M_1)^\sharp = \Pi^{-1} M_1,$$

where the \sharp denotes the dual with respect to the hermitian form h .

Recall that x is said to be “0-critical” in the sense of [BC] if and only if $\Pi M_0 = V M_0$, which is equivalent to the relation $M_0^\sharp = M_0$, i.e. $x \in \mathbb{P}_\Lambda(\mathbb{F})$ for a vertex lattice Λ of type 0.

Similarly x is “1-critical” if and only if $\Pi M_1 = V M_1$, which is equivalent to the relation $((V M_1)^\sharp)^\sharp = p(V M_1)^\sharp$. This last condition is then equivalent to the condition $x \in \mathbb{P}_{\Lambda'}(\mathbb{F})$ for a vertex lattice Λ' of type 2.

In particular, this discussion implies that our use of the terms “ordinary”, “special” and “super-special” coincides with their use in [KR1, §1]. \diamond

2.2. Unitary special cycles. We begin this section by defining the *(local) unitary special cycles*, whose construction is due to Kudla and Rapoport [KR2].

To start, fix a triple $\underline{\mathbb{Y}} = (\mathbb{Y}, i_{\mathbb{Y}}, \lambda_{\mathbb{Y}})$ over \mathbb{F} consisting of

- (i) a (supersingular) p -divisible group \mathbb{Y} of dimension 1 and height 2 over \mathbb{F} ;
- (ii) an action $i_{\mathbb{Y}} : o_{k,p} \rightarrow \text{End}(\mathbb{Y})$ such that on $\text{Lie}(\mathbb{Y})$, this action coincides with the action of $o_{k,p}$ via the embedding $\tau_0 : o_{k,p}/(p) \rightarrow \mathbb{F}$;
- (iii) and finally, a principal polarization $\lambda_{\mathbb{Y}}$ such that the induced Rosati involution $*$ satisfies

$$i_{\mathbb{Y}}(a)^* = i_{\mathbb{Y}}(a'), \quad \text{for all } a \in o_{k,p}.$$

We define two spaces of *special homomorphisms*:

$$(2.8) \quad \mathbb{V}_\phi^+ := \{\mathbf{b} \in \text{Hom}(\mathbb{Y}, \mathbb{X}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \mid \mathbf{b} \circ i_{\mathbb{Y}}(a) = \iota_{\mathbb{X}}(\phi(a)) \circ \mathbf{b} \text{ for all } a \in o_{k,p}\}$$

and

$$(2.9) \quad \mathbb{V}_\phi^- := \{\mathbf{b} \in \text{Hom}(\mathbb{Y}, \mathbb{X}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \mid \mathbf{b} \circ i_{\mathbb{Y}}(a) = \iota_{\mathbb{X}}(\phi(a')) \circ \mathbf{b} \text{ for all } a \in o_{k,p}\}.$$

Using the polarization $\lambda_{\mathbb{Y}}$, and the polarization $\lambda_{\mathbb{X},\phi}$ as defined in (2.3), we may construct natural hermitian forms h^+ and h^- on \mathbb{V}_{ϕ}^+ and \mathbb{V}_{ϕ}^- respectively; these are defined by the formulas

$$(2.10) \quad h^+(\mathbf{b}_1, \mathbf{b}_2) := \lambda_{\mathbb{Y}}^{-1} \circ \mathbf{b}_2^{\vee} \circ \lambda_{\mathbb{X},\phi} \circ \mathbf{b}_1 \in \text{End}_{o_k}(\underline{\mathbb{Y}}^+) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq k_p$$

$$(2.11) \quad h^-(\mathbf{b}_1, \mathbf{b}_2) := \lambda_{\mathbb{Y}}^{-1} \circ \mathbf{b}_1^{\vee} \circ \lambda_{\mathbb{X},\phi} \circ \mathbf{b}_2 \in \text{End}_{o_k}(\underline{\mathbb{Y}}^+) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq k_p.$$

Let $q^{\pm}(\mathbf{b}) := h^{\pm}(\mathbf{b}, \mathbf{b})$ denote the corresponding quadratic forms.

Our next step is to relate the spaces \mathbb{V}_{ϕ}^{\pm} to the hermitian space (C, h) . Let $M(\mathbb{Y})$ denote the Dieudonné module over $W = W(\mathbb{F})$ attached to \mathbb{Y} ; this is a free W -module of rank 2. As before, we have a grading $M(\mathbb{Y}) = M(\mathbb{Y})_0 \oplus M(\mathbb{Y})_1$, where

$$M(\mathbb{Y})_i := \{m \in M(\mathbb{Y}) \mid i_{\mathbb{Y}}(a) \cdot m = \tau_i(a)m, \text{ for all } a \in o_{k,p}\}.$$

Moreover, we may choose generators f_0 and f_1 for $M(\mathbb{Y})_0$ and $M(\mathbb{Y})_1$ respectively, such that $Vf_0 = f_1, Vf_1 = pf_0$, and that the alternating form $\{\cdot, \cdot\}_{\mathbb{Y}}$ defined by the polarization $\lambda_{\mathbb{Y}}$ satisfies

$$(2.12) \quad \{f_0, f_1\}_{\mathbb{Y}} = \delta;$$

see [KR2, Remark 2.5].

Suppose that $\mathbf{b} \in \mathbb{V}_{\phi}^+$. Abusing notation, we denote the corresponding map on (rational) Dieudonné modules again by

$$\mathbf{b} : N(\mathbb{Y}) = M(\mathbb{Y}) \otimes \mathbb{Q} \rightarrow N(\mathbb{X}).$$

Let $b := \mathbf{b}(f_0)$. Then since \mathbf{b} is $o_{k,p}$ -linear, we have that $b \in N(\mathbb{X})_0$, and furthermore,

$$V^{-1}Fb = pV^{-2}b = p\mathbf{b}(V^{-2}f_0) = b$$

so $b \in C$. Finally, we note that

$$\mathbf{b}(f_1) = \mathbf{b}(Vf_0) = V\mathbf{b}(f_0) = Vb,$$

and so \mathbf{b} is determined by b . We therefore obtain an isomorphism

$$(2.13) \quad \varphi^+ : \mathbb{V}_{\phi}^+ \rightarrow C, \quad \mathbf{b} \mapsto b := \mathbf{b}(f_0).$$

It is straightforward to show that $q(\varphi^+\mathbf{b}) = p^{-1}q^+(\mathbf{b})$, where q and q^+ are the quadratic forms on C and \mathbb{V}_{ϕ}^+ defined by (2.6) and (2.10) respectively.

In a similar manner, if $\mathbf{b} \in \mathbb{V}_{\phi}^-$, then $b := \mathbf{b}(f_1) \in C$, and \mathbf{b} is again determined by b . Hence we obtain an isomorphism

$$(2.14) \quad \varphi^- : \mathbb{V}_{\phi}^- \rightarrow C, \quad \mathbf{b} \mapsto b := \mathbf{b}(f_1)$$

such that $q(\varphi^-\mathbf{b}) = q^-(\mathbf{b})$.

Finally, we come to the definition of the unitary special cycles, as in [KR2]. Note that the p -divisible group $\underline{\mathbb{Y}} = (\mathbb{Y}, i_{\mathbb{Y}})$, together with its $o_{k,p}$ -action, admits a canonical lift $\underline{\mathbb{Y}}_W = (\mathbb{Y}_W, i_{\mathbb{Y}_W})$ to W . For any $S \in \mathbf{Nilp}$, we let $\underline{\mathbb{Y}}_S = (\mathbb{Y}_S, i_{\mathbb{Y}_S})$ denote the base change $\mathbb{Y}_S = \mathbb{Y}_W \times_W S$, with induced $o_{k,p}$ -action.

Definition 2.5. For $\mathbf{b} \in \mathbb{V}_{\phi}^{\pm}$, we define the local unitary special cycle $Z(\mathbf{b})$ by the following moduli problem: for $S \in \mathbf{Nilp}$, let $Z(\mathbf{b})(S)$ denote the set of points $(X, \iota_X, \rho) \in \mathcal{D}(S)$ such that quasi-isogeny

$$\rho^{-1} \circ \mathbf{b} : \mathbb{Y} \times_{\mathbb{F}} \overline{S} \rightarrow X \times_S \overline{S}$$

lifts to a morphism of p -divisible groups $\mathbb{Y}_S \rightarrow X$.

Remark 2.6. (i) If such a lift exists, then it is unique, by rigidity for p -divisible groups. In particular, if the lift exists, then it is $o_{k,p}$ -linear (resp. $o_{k,p}$ -antilinear) whenever $\mathbf{b} \in \mathbb{V}_\phi^+$ (resp. $\mathbf{b} \in \mathbb{V}_\phi^-$).

(ii) By [RZ, Proposition 2.9], the moduli problem $Z(\mathbf{b})$ is represented by a *closed formal subscheme* of \mathcal{D} . \diamond

Our first result is the following description of the \mathbb{F} -points of the special cycles.

Proposition 2.7. (i) Suppose $\mathbf{b} \in \mathbb{V}_\phi^-$, and $b = \varphi^-\mathbf{b} \in C$ is the corresponding vector, cf. (2.14). Let Λ be any vertex lattice. Then

$$Z(\mathbf{b})(\mathbb{F}) \cap \mathbb{P}_\Lambda(\mathbb{F}) = \begin{cases} \emptyset, & \text{if } b \notin \Lambda, \\ a \text{ single point } \{x\}, & \text{if } b \in \Lambda - p\Lambda, \\ \mathbb{P}_\Lambda(\mathbb{F}), & \text{if } b \in p\Lambda. \end{cases}$$

If the second case above occurs, then Λ is necessarily of type 0, and the point x is special. Moreover x is superspecial if and only if $\text{ord}_p q^-(\mathbf{b}) > 0$.

(ii) Similarly, for $\mathbf{b} \in \mathbb{V}_\phi^+$ with $b = \varphi^+\mathbf{b}$, cf. (2.13), we have

$$Z(\mathbf{b})(\mathbb{F}) \cap \mathbb{P}_\Lambda(\mathbb{F}) = \begin{cases} \emptyset, & \text{if } b \notin \Lambda, \\ a \text{ single point } \{x\}, & \text{if } b \in \Lambda - p\Lambda, \text{ with } \Lambda^\sharp = p\Lambda \\ \mathbb{P}_\Lambda(\mathbb{F}), & \text{if } b \in \Lambda \text{ with } \Lambda^\sharp = \Lambda, \\ \mathbb{P}_\Lambda(\mathbb{F}), & \text{if } b \in p\Lambda, \Lambda \text{ arbitrary.} \end{cases}$$

In the second case, the unique point x is special, and is superspecial if and only if $\text{ord}_p q^+(\mathbf{b}) > 0$.

Proof. (i) Suppose $\mathbf{b} \in \mathbb{V}_\phi^-$. We observe that a point $x = (\underline{X}, \rho_X) \in \mathcal{D}(\mathbb{F})$ is in $Z(\mathbf{b})(\mathbb{F})$ if and only if, upon identifying $M(X)$ with a W -lattice in $N(\mathbb{X})$, we have

$$\begin{aligned} \mathbf{b}(M(\mathbb{Y})) \subset M(X) &\iff \mathbf{b}(f_0) \in M(X)_1, \text{ and } \mathbf{b}(f_1) \in M(X)_0 \\ &\iff b = \mathbf{b}(f_1) \in VM(X)_1; \end{aligned}$$

the last equivalence follows from the relation $V\mathbf{b}(f_0) = \mathbf{b}(f_1)$. Recall also that $\mathbf{b}(f_0) \in C = N(\mathbb{X})_0^{FV^{-1}}$, so we have

$$x = (\underline{X}, \rho_X) \in Z(\mathbf{b})(\mathbb{F}) \iff b \in VM(X)_1 \cap C.$$

Now suppose $x \in \mathbb{P}_\Lambda(\mathbb{F}) \cap Z(\mathbf{b})(\mathbb{F})$, with $\Lambda^\sharp = p\Lambda$. By construction, x corresponds to a lattice pair $B \subset A$, where

$$A = VM(X)_1^\sharp = \Lambda \otimes_{\tau_0} W.$$

On the other hand, note

$$pA = A^\sharp = (VM_1(X)^\sharp)^\sharp = FM_1(X).$$

Thus, as $x \in Z(\mathbf{b})(\mathbb{F})$ as well, then

$$b \in VM_1(X) \cap C = FM_1(X) \cap C = pA \cap C.$$

However, as $pA = p\Lambda_W$, the above line is true for *all* $x \in \mathbb{P}_\Lambda(\mathbb{F})$, as soon as it is true for a single point. Hence we have $\mathbb{P}_\Lambda(\mathbb{F}) \subset Z(\mathbf{b})(\mathbb{F})$.

Now suppose $x \in \mathbb{P}_\Lambda(\mathbb{F})$ with $\Lambda^\# = \Lambda$. By construction, this means $M(X)_0$ is identified with $\Lambda \otimes_{\tau_0} W$, and so if $\mathbb{P}_\Lambda(\mathbb{F}) \cap Z(\mathbf{b}) \neq \emptyset$, then we must have $b \in \Lambda$. Furthermore, any $x \in \mathbb{P}_\Lambda(\mathbb{F})$ is determined by the sequence of inclusions of \mathbb{F} -codimension 1

$$\begin{array}{ccccc} pM(X)_0 & \subset & VM(X)_1 & \subset & M(X)_0 \\ \parallel & & & & \parallel \\ p\Lambda \otimes W & & & & \Lambda \otimes W. \end{array}$$

Hence, if $b \in p\Lambda$, then $b \in VM(X)_1$ for all $(\underline{X}, \rho_X) \in \mathbb{P}_\Lambda(\mathbb{F})$, and so

$$Z(\mathbf{b})(\mathbb{F}) \cap \mathbb{P}_\Lambda(\mathbb{F}) = \mathbb{P}_\Lambda(\mathbb{F}).$$

If on the other hand $b \in \Lambda \setminus p\Lambda$, and $\mathbb{P}_\Lambda(\mathbb{F}) \cap Z(\mathbf{b})(\mathbb{F}) \neq \emptyset$, then this intersection necessarily consists of a single point $x = (X, \rho_X)$: namely, the unique point with

$$VM(X)_1 = W \cdot b + p\Lambda_W \subset \Lambda_W.$$

Note in this case, $(pV^{-2})VM(X)_1 = VM(X)_1$ as both Λ_W and b are pV^{-2} -invariant.

Now by construction, we have

$$\Lambda_W \subset VM(X)_1^\# \subset p^{-1}\Lambda_W.$$

If $\text{ord}_p q(b) = 0$, then $p^{-1}b \notin VM(X)_1^\#$, and so the point is ordinary. On the other hand, if $\text{ord}_p q(b) > 0$, then

$$VM(X)_1^\# = W \cdot p^{-1}b + \Lambda_W = p^{-1}VM(X)_1 = p^{-1} \left(VM(X)_1^\# \right),$$

and so this point is superspecial.

(ii) Suppose $\mathbf{b} \in \mathbb{V}_\phi^+$, and let $b = \mathbf{b}(f_0) = \varphi^+(\mathbf{b})$. A point $x = (\underline{X}, \rho_X)$ lies in $Z(\mathbf{b})(\mathbb{F})$ if and only if $b \in M(X)_0$. The lemma follows via a similar argument to the previous case. \square

Lemma 2.8. *Fix an ϵ -invariant basis $\{v_0, v_1\}$ of C (cf. Remark 2.4) such that*

$$h(v_0, v_0) = h(v_1, v_1) = 0, \quad h(v_0, v_1) = -h(v_1, v_0) = \delta.$$

Suppose that $b = a_0v_0 + a_1v_1$, with $\text{ord}_p q(b)$ either 0 or -1 . Set $b' := \epsilon(b) = a'_0v_0 + a'_1v_1$ and

$$\Lambda_b := \text{span}_{o_{k,p}} \{b, b'\}.$$

If $\text{ord}_p q(b) = 0$, then Λ_b is of type 0 (i.e. $\Lambda_b^\# = \Lambda_b$), and is the unique lattice of type 0 such that $b \in \Lambda_b - p\Lambda_b$. Similarly, if $\text{ord}_p q(b) = -1$, then Λ_b is of type 2, and is the unique lattice of type 2 such that $b \in \Lambda_b - p\Lambda_b$.

Proof. Suppose that $\text{ord}_p q(b) = 0$, and Λ is a type 0 lattice with $b \in \Lambda - p\Lambda$. Then there exists an element $\gamma \in SU(C)$ such that

$$\gamma \cdot \Lambda = \Lambda_b,$$

as $SU(C)$ acts transitively on the set of type 0 lattices. We may assume without loss of generality that $q(b) = 1$. Note the vectors b and b' form an orthogonal basis for C , with $h(b, b) = -h(b', b') = 1$. Let

$$[\gamma] = \begin{pmatrix} x & y \\ w & z \end{pmatrix}$$

denote the matrix representation of γ with respect to the basis $\{b, b'\}$. Since $b = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \Lambda$, we have $x, w \in o_{k,p}$. On the other hand, the equation $\gamma \cdot \gamma^* = 1$ implies

$$xx' - yy' = -ww' + zz' = 1, \quad yw' = xz',$$

which implies that $y, w \in o_{k,p}$ as well. Hence γ and γ^* stabilize Λ_b , which yields the result $\Lambda = \Lambda_b$.

The proof in the case $\text{ord}_p q(b) = -1$ is similar. \square

Remark 2.9. Suppose $\mathbf{b} \in \mathbb{V}_\phi^\pm$, with $\text{ord}_p q^\pm \mathbf{b} = 0$. Let $b = \varphi^\pm \mathbf{b} \in C$ denote the corresponding vector. Then by Lemma 2.8, there is a *unique* lattice Λ_\odot such that $b \in \Lambda_\odot - p\Lambda_\odot$, and Λ_\odot is of type 0 (resp. of type 2) if $\mathbf{b} \in \mathbb{V}_\phi^-$ (resp. $\mathbf{b} \in \mathbb{V}_\phi^+$). Combining this observation with Proposition 2.7, we find that

$$Z(\mathbf{b})(\mathbb{F}) = \{x\}$$

is a *single ordinary point* lying in the component $\mathbb{P}_{\Lambda_\odot}(\mathbb{F})$. \diamond

Our goal for the remainder of this section is to give a complete description of the special cycles $Z(\mathbf{b})$ as cycles on \mathcal{D} , as in Theorem 2.14 below. We shall do this by writing down equations using the (formal) affine open cover described in [KR1, §1], which consist of affine schemes of two types:

- (i) $\widehat{\Omega}_\Lambda^{\text{ord}} \simeq \text{Spf}W[T, (T^p - T)^{-1}]^\vee$, for each vertex lattice Λ , and
- (ii) $\widehat{\Omega}_{[\Lambda, \Lambda']} \simeq \text{Spf}W[T_1, T_2, (T_1^{p-1} - 1)^{-1}(T_2^{p-1} - 1)^{-1}]^\vee$, for each pair of neighbouring vertex lattices Λ and Λ' .

In both cases above, the superscript $^\vee$ denotes completion along the ideal generated by p , and the isomorphisms are determined by a choice of basis for Λ . The underlying set of $\widehat{\Omega}_\Lambda^{\text{ord}}$ is the set of ordinary points $\mathbb{P}_\Lambda^{\text{ord}}$ in \mathbb{P}_Λ (that is, the complement of the superspecial points), while the underlying set of $\widehat{\Omega}_{[\Lambda, \Lambda']}$ is the union

$$\mathbb{P}_\Lambda^{\text{ord}} \cup \mathbb{P}_{\Lambda'}^{\text{ord}} \cup \{x\}$$

where x is the superspecial point at the intersection of $\mathbb{P}_\Lambda \cap \mathbb{P}_{\Lambda'}$. For Λ a type 0 lattice, with neighbour Λ' of type 2, we have open immersions (cf. [KR1, §1]),

$$\widehat{\Omega}_\Lambda^{\text{ord}} \rightarrow \widehat{\Omega}_{[\Lambda, \Lambda']}, \quad \text{induced by } T_0 \mapsto T, \ T_1 \mapsto pT^{-1}$$

and

$$\widehat{\Omega}_{\Lambda'}^{\text{ord}} \rightarrow \widehat{\Omega}_{[\Lambda, \Lambda']}, \quad \text{induced by } T_0 \mapsto pT^{-1}, \ T_1 \mapsto T.$$

We first consider a type 0 lattice $\Lambda = \Lambda^\sharp$. Let $\mathbf{b} \in \mathbb{V}_\phi^\pm$, with $\text{ord}_p q^\pm(\mathbf{b}) \geq 0$ and such that $Z(\mathbf{b}) \cap \widehat{\Omega}_\Lambda^{\text{ord}} \neq \emptyset$. Let $b = \varphi^\pm \mathbf{b} \in C$ the corresponding vector; by Proposition 2.7, we must have $b \in \Lambda$. Fix an ϵ -invariant basis $\{v_0, v_1\}$ of Λ , such that with respect to this basis, the hermitian form h has matrix

$$(2.15) \quad h \sim \begin{pmatrix} & \delta \\ -\delta & \end{pmatrix},$$

and write

$$(2.16) \quad b = a_0 v_0 + a_1 v_1,$$

where $a_0, a_1 \in o_{k,p}$.

Proposition 2.10. *Let $\mathbf{b} \in \mathbb{V}_\phi^\pm$, and suppose $Z(\mathbf{b}) \cap \widehat{\Omega}_\Lambda^{\text{ord}} \neq \emptyset$. Then, with notation as in the previous paragraph,*

$$Z(\mathbf{b}) \cap \widehat{\Omega}_\Lambda^{\text{ord}} \simeq \text{Spf}W[T, (T^p - T)^{-1}]^\vee / (f),$$

where

$$(2.17) \quad f := \begin{cases} a_0 T + a_1, & \text{if } \mathbf{b} \in \mathbb{V}_\phi^- \\ p(a'_0 T + a'_1), & \text{if } \mathbf{b} \in \mathbb{V}_\phi^+. \end{cases}$$

Proof. The proof of this proposition is modelled on arguments by Terstiege found in the proofs of [Ter, Propositions 2.8 and 4.5]. We consider the following cases:

Case 1: $\mathbf{b} \in \mathbb{V}_\phi^-$, with $b = \varphi^- \mathbf{b} \in \Lambda$.

Fix a point $x \in Z(\mathbf{b})(\mathbb{F}) \cap \mathbb{P}_\Lambda(\mathbb{F})$, corresponding to $(\overline{X}, \iota_{\overline{X}}, \rho) \in \mathcal{D}(\mathbb{F})$. At present, we do not require that x be ordinary (i.e. \overline{X} may be superspecial); in fact the argument we are about to discuss applies equally well in the superspecial case, and so it will be expedient to consider this slightly more general setting here. Consider the (complete) local ring

$$R = \mathcal{O}_{\mathcal{D}, x}.$$

Let m_R denote the maximal ideal of R , and I denote the ideal corresponding to $Z(\mathbf{b})$. For the purposes of the proposition, we need to prove that I is generated by the image of f in R . Note R is Noetherian, which implies that I is complete for the p -adic topology. Hence if we set

$$R_n := R/p^n R, \quad I_n = I + p^n R \subset R_n,$$

it will suffice to prove that I_n is generated by the image of f in R_n , for each n . Let m_n denote the maximal ideal of R_n . We set

$$A := R_n/m_n I_n, \quad A' := R_n/I_n.$$

Then the kernel $J := I_n/m_n I_n$ of the projection $A \rightarrow A'$ satisfies $J^2 = 0$, and hence is endowed with a PD structure. Moreover, to prove the proposition, it suffices (by Nakayama) to show that J is generated by the image of f in A (which, abusing notation, we shall henceforth denote as f).

Finally, we note that both A and A' can be viewed as $o_{k,p}$ -algebras via the fixed embedding $\tau_0 : o_{k,p} \rightarrow W$ composed with the respective structural morphisms.

Now associated to the rings A and A' are two points in the moduli space \mathcal{D} , which in turn correspond to formal \mathcal{O}_B -modules X and X' , both of whose special fibres are equal to \overline{X} . Moreover, by assumption, X' is in $Z(\mathbf{b})(A')$. In other words, the map

$$\beta_{\overline{X}} := \rho^{-1} \circ \mathbf{b} : \mathbb{Y} \rightarrow \overline{X}$$

is a morphism of p -divisible groups, which lifts to a morphism

$$(2.18) \quad \beta_{X'} : \mathbb{Y}_{A'} \rightarrow X'.$$

Here $\mathbb{Y}_{A'}$ is the base change of the canonical lift \mathbb{Y}_W to A' .

Let $\mathbb{D}(\mathbb{Y}_{A'}/\cdot)$ denote the Grothendieck-Messing crystal of $\mathbb{Y}_{A'}$, which carries a $\mathbb{Z}/2\mathbb{Z}$ grading induced by the action of $o_{k,p}$. In particular, for a PD extension $B \rightarrow A'$, the canonical lift of $\mathbb{Y}_{A'}$ (together with its $o_{k,p}$ -action) to B is determined by the Hodge filtration:

$$(2.19) \quad \mathbb{D}(\mathbb{Y}_{A'}/B)_1 = \mathcal{F}_{\mathbb{Y}_B} \subset \mathbb{D}(\mathbb{Y}_{A'}/B).$$

This is a consequence of the requirement that $o_{k,p}$ act on $Lie(\mathbb{Y}_B)$ with signature $(1, 0)$.

Turning to the p -divisible groups X and X' , we let $\mathbb{D}(X/\cdot)$ and $\mathbb{D}(X'/\cdot)$ denote their respective Grothendieck-Messing crystals. We then have a diagram

$$(2.20) \quad \begin{array}{ccc} \mathbb{D}(X/A) & \simeq & \mathbb{D}(X'/A) \\ & & \downarrow \text{mod } J \\ & & \mathbb{D}(X'/A') \end{array}$$

Let $B \rightarrow A'$ be a PD-extension as above. Since $X' \in Z(\mathbf{b})(A')$, there is a map of B -modules

$$\mathbb{D}(\beta_{X'}): \mathbb{D}(\mathbb{Y}_{A'}/B) \rightarrow \mathbb{D}(X'/B)$$

induced by (2.18). By Grothendieck-Messing theory, if \tilde{X} is a lift of X' to B which corresponds to an $\mathcal{O}_{B,p}$ -stable direct summand $\mathcal{F} \subset \mathbb{D}(X'/B)$, then

$$\tilde{X} \in Z(\mathbf{b})(B) \iff \mathbb{D}(\beta_{X'}) (\mathcal{F}_{\mathbb{Y}_B}) \subset \mathcal{F}.$$

Fixing any basis vector $f_B \in \mathcal{F}_{\mathbb{Y}_B} = \mathbb{D}(\mathbb{Y}_{A'}/B)_1$, and recalling that \mathbf{b} is $o_{k,p}$ -antilinear, the above condition can be rewritten as

$$(2.21) \quad \tilde{X} \in Z(\mathbf{b})(B) \iff \mathbb{D}(\beta_{X'})(f_B) \in \mathcal{F}_0.$$

For convenience, we fix a basis vector $f'_1 \in \mathbb{D}(\mathbb{Y}_{A'}/A')_1$, and a lift to a basis vector $f_1 \in \mathbb{D}(\mathbb{Y}_A/A)_1$.

We want to express the condition (2.21) in terms of the coordinate chart

$$(2.22) \quad \hat{\Omega}_\Lambda^{ord} \simeq Spf W[T, (T^p - T)^{-1}]^\vee,$$

when \overline{X} is an ordinary point; note that as p is nilpotent in A , the point $X \in \hat{\Omega}_\Lambda^{ord}(A)$ is determined by a map of W -algebras

$$(2.23) \quad W[T, (T^p - T)^{-1}] \rightarrow A,$$

which in turn is determined simply by an element $t \in A$, with $t^p - t \in A^\times$. In particular, $t \in A^\times$. More generally, as we wish to include superspecial points in this analysis, we may suppose $\overline{X} \in \mathbb{P}_\Lambda(\mathbb{F})$ is a point in a neighbourhood

$$(2.24) \quad \hat{\Omega}_{[\Lambda, \Lambda']} \simeq Spf W[T_0, T_1, (T_0^{p-1} - 1)^{-1}, (T_1^{p-1} - 1)^{-1}] / (T_0 T_1 - p)^\vee$$

of a super-special point; let $t \in A$ denote the image of T_0 in A corresponding to the point $X \in \hat{\Omega}_{[\Lambda, \Lambda']}(A)$, so that $t^{p-1} - 1 \in A^\times$. We claim that we may then find isomorphisms

$$(2.25) \quad \mathbb{D}(X/A)_0 \simeq \Lambda \otimes_{o_{k,p}} A, \quad \mathbb{D}(X'/A')_0 \simeq \Lambda \otimes_{o_{k,p}} A'$$

such that

- (i) the 0-th graded piece of the Hodge filtration for X is identified with

$$(\mathcal{F}_X)_0 = span_A \{v_0 \otimes 1 - v_1 \otimes t\} \subset \mathbb{D}(X/A)_0 \simeq \Lambda \otimes A,$$

where $\{v_0, v_1\}$ is the fixed basis of Λ as in (2.15);

- (ii) the element $\mathbb{D}(\beta_{X'})(f_1)$ is identified with $b \otimes 1 \in \Lambda \otimes A$;
- (iii) the vertical map in (2.20) is the natural map

$$\Lambda \otimes A \rightarrow \Lambda \otimes A', \quad l \otimes a \mapsto l \otimes a',$$

where $a \mapsto a'$ is the projection $A \rightarrow A'$.

This essentially follows by carefully tracing through the construction of the coordinate charts (2.23) and (2.24) as described in [BC]. The idea is that a point $(X, \iota_X, \rho_X) \in \mathcal{D}(A)$ meeting the component \mathbb{P}_Λ determines the following data: (i) a constant sheaf η_0 of \mathbb{Z}_p -modules on $Spec(A)$; (ii) a surjective map

$$u_0 : \eta_0 \otimes A \rightarrow Lie(X)_0,$$

where $Lie(X)_0$ is the 0-graded component of the Lie algebra of X ; and (iii) an isomorphism

$$r_0 : \eta_0 \otimes_{\mathbb{Z}_p} o_{k,p} \xrightarrow{\sim} \underline{\Lambda},$$

where $\underline{\Lambda}$ is viewed as the constant sheaf on $\text{Spec}(S)$. This data satisfies some compatibility conditions, cf. [BC, §II.8]. In particular, if \overline{X} is the special fibre of X , with Dieudonné module $\overline{M} = \overline{M}_0 \oplus \overline{M}_1$, we have an identification (Proposition II.3.14 of loc. cit.)

$$\eta_0 \otimes_{o_{k,p}} \simeq (\overline{M}_0)^{V^{-1}F}$$

such that r_0 is simply the isomorphism induced by the framing $\rho : \overline{X} \rightarrow \mathbb{X}$.

Finally, recalling the fixed ϵ -invariant basis $\{v_0, v_1\}$ for Λ , the point X corresponds to the coordinate $t \in A$ as above if and only if we have an exact sequence of A -modules as follows.

$$(2.26) \quad 0 \longrightarrow \text{span}_A(v_0 \otimes 1 - v_1 \otimes t) \longrightarrow \eta_0 \otimes_{\mathbb{Z}_p} A \xrightarrow{u_0} \text{Lie}(X)_0 \longrightarrow 0$$

$$\quad \quad \quad \parallel \quad r_0$$

$$\quad \quad \quad \Lambda \otimes_{o_{k,p}} A$$

On the other hand, we have a canonical short exact sequence

$$(2.27) \quad 0 \longrightarrow (\mathcal{F}_X)_0 \longrightarrow \mathbb{D}(X/A)_0 \longrightarrow \text{Lie}(X)_0 \longrightarrow 0$$

from Grothendieck-Messing theory. Hence, any isomorphism

$$\psi : \eta_0 \otimes A \rightarrow \mathbb{D}(X/A)_0$$

that preserves $\text{Lie}(X)_0$ yields an isomorphism as in (2.25) which satisfies conditions (i) and (ii). If ψ is any such isomorphism, it may be modified so that condition (iii) also holds. Indeed, let $b^\circ = \psi(b \otimes 1)$. Then the images of b° and $\mathbb{D}(\beta_{X'})(f_1)$ in $\text{Lie}(X)_0$ coincide, and so we can replace ψ by

$$\tilde{\psi} = \psi + g, \quad \text{for some } g \in \text{Hom}(\eta_0 \otimes A, (\mathcal{F}_X)_0)$$

so that $\tilde{\psi}(b \otimes 1) = \mathbb{D}(\beta_{X'})(f_1)$ as required.

Now suppose \mathfrak{b} is any ideal of A contained in J , and let $B := A/\mathfrak{b}$. Let t_B denote the image of t in B and X_B the corresponding B -valued point of \mathcal{D} . We may view X_B as a deformation of X' , which by Grothendieck-Messing theory corresponds to the direct summand

$$\mathcal{F}_{X_B} = \mathcal{F}_X \otimes_A B \subset \mathbb{D}(X'/B),$$

and in particular we have

$$(\mathcal{F}_{X_B})_0 = \text{span}_B\{v_0 \otimes 1 - v_1 \otimes t_B\} \subset \mathbb{D}(X'/B)_0 = \Lambda \otimes B.$$

Then in light of (2.21), and noting that \mathfrak{b} is $o_{k,p}$ -antilinear, we have

$$\begin{aligned} X_B \in Z(\mathfrak{b})(B) &\iff \mathbb{D}(\beta_{X'}/B)(f_1) \otimes 1 \in (\mathcal{F}_{\tilde{X}})_0 \\ &\iff b \otimes 1 \in \text{span}\{v_0 \otimes 1 - v_1 \otimes t_B\} \text{ in } \Lambda \otimes B. \end{aligned}$$

Recall that we had written $b = a_0 v_0 + a_1 v_1$. Hence the last condition is equivalent to

$$a_0 t_B + a_1 = 0 \text{ in } B.$$

Applying the necessity of this condition to the case $\mathfrak{b} = J$, so $B = A/J$, we find that

$$f = a_0 t + a_1 \in J.$$

On the other hand, by the sufficiency of the above condition, the map $\text{Spf}(A/(f)) \rightarrow \widehat{\Omega}_\Lambda^{\text{ord}}$ factors through $Z(\mathfrak{b})$. But by definition J is the smallest ideal of A such that $\text{Spf}(A/J) \rightarrow \widehat{\Omega}_\Lambda^{\text{ord}}$ factors through $Z(\mathfrak{b})$, and so $J = (f)$; under the hypotheses of the proposition (i.e. \overline{X} is ordinary), this concludes the proof.

Case 2: $\mathbf{b} \in \mathbb{V}_\phi^+$ with $b = \varphi^+ \mathbf{b} \in \Lambda$.

Let $r = r(b, \Lambda) := \max\{n \mid p^{-n}b \in \Lambda\}$, and write

$$b = p^r b_0 = p^r(\alpha_0 v_0 + \alpha_1 v_1),$$

where $b_0 = \alpha_0 v_0 + \alpha_1 v_1 \in \Lambda - p\Lambda$, and so in particular at least one of α_0, α_1 is a unit in $\mathcal{O}_{k,p}^\times$.

As a first step, we shall prove that every $W_{r+1} := W/(p^{r+1})$ valued point of $\widehat{\Omega}_\Lambda^{ord}$ belongs to the special cycle $Z(\mathbf{b})$. To this end, suppose $(X, \rho_X) \in \widehat{\Omega}_\Lambda^{ord}(W_{r+1})$, and let $\overline{X} \in \Omega_\Lambda^{ord}(\mathbb{F})$ denote its reduction modulo p . Let

$$\overline{M} = M(\overline{X}) = \overline{M}_0 \oplus \overline{M}_1$$

denote the Dieudonné module of \overline{X} , endowed with the grading induced by the action of $\mathcal{O}_{k,p}$. As the projection $W_{r+1} \rightarrow \mathbb{F}$ has kernel generated by p , it is equipped with a PD structure. Hence, via Grothendieck-Messing theory, the point X corresponds to an $\mathcal{O}_{B,p}$ -stable summand

$$\mathcal{F} \subset \mathbb{D}(\overline{X}/W_{r+1}) = \overline{M} \otimes_W W_{r+1} = \overline{M}/p^{r+1}\overline{M},$$

such that

$$\mathcal{F} \otimes_{W_{r+1}} \mathbb{F} = V\overline{M}/p\overline{M}.$$

By Proposition 2.7, we have $\overline{X} \in Z(\mathbf{b})(\mathbb{F})$, and so the map $\beta := \rho_X^{-1} \circ \mathbf{b} : \mathbb{Y} \rightarrow \overline{X}$ induces a morphism of Dieudonné modules

$$\beta : M(\mathbb{Y}) = W \cdot f_0 \oplus W \cdot f_1 \rightarrow \overline{M}.$$

By definition, we have

$$\beta(f_0) = b \in \Lambda \otimes W = \overline{M}_0, \quad \beta(f_1) = V\beta(f_0) = Vb \in \overline{M}_1.$$

The morphism β also induces a map on crystals

$$\mathbb{D}(\beta/W_{r+1}) : M(\mathbb{Y}) \otimes_W W_{r+1} = \mathbb{D}(\mathbb{Y}/W_{r+1}) \rightarrow \mathbb{D}(\overline{X}/W_{r+1}) = \overline{M} \otimes_W W_{r+1}.$$

Recall that the direct summand

$$\mathcal{F}_\mathbb{Y} \subset M(\mathbb{Y}) \otimes_W W_{r+1}$$

corresponding to the lift $\mathbb{Y}_{W_{r+1}}$ of \mathbb{Y} is simply the rank-1 module $\mathcal{F}_\mathbb{Y} = \text{span}_{W_{r+1}}\{f_1 \otimes 1\}$. Hence, by Grothendieck-Messing, we have

$$\begin{aligned} X \in Z(\mathbf{b})(W_{r+1}) &\iff \mathbb{D}(\beta/W_{r+1})(f_1) \in \mathcal{F}_1 \\ (2.28) \quad &\iff (Vb) \otimes 1 \in \mathcal{F}_1 \text{ in } \overline{M}_1 \otimes W_{r+1}. \end{aligned}$$

We shall show that this latter condition always holds for any $X \in \widehat{\Omega}_\Lambda^{ord}(W_{r+1})$.

Consider the element $b_0 = p^{-r}b \in \Lambda \setminus p\Lambda$. If $Vb_0 \in p\overline{M}_1$, then

$$Vb \otimes 1 = p^r \cdot Vb_0 \otimes 1 \in p^{r+1} \cdot (\overline{M}_1 \otimes W_{r+1}) = \{0\},$$

and so (2.28) holds trivially. If on the other hand $Vb_0 \in \overline{M}_1 - p\overline{M}_1$, then

$$\text{span}_\mathbb{F}(Vb_0 + p\overline{M}_1) = V\overline{M}_0/p\overline{M}_1 = \mathcal{F}_1 \otimes_{W_{r+1}} \mathbb{F};$$

in other words, the image of Vb_0 in $\overline{M}_1/p\overline{M}_1$ is a basis vector for $\mathcal{F}_1 \otimes \mathbb{F}$. Hence, there exists $\alpha \in \overline{M}_1 \otimes W_{r+1}$ such that

$$Vb_0 \otimes 1 + p\alpha \in \mathcal{F}_1,$$

and so

$$p^r(Vb_0 \otimes 1) + p^{r+1}\alpha = Vb \otimes 1 \in \mathcal{F}_1,$$

as required. Thus, we have proven

$$(2.29) \quad Z(\mathbf{b}) \cap \widehat{\Omega}_{\Lambda}^{ord}(W_{r+1}) = \widehat{\Omega}_{\Lambda}^{ord}(W_{r+1}).$$

We now determine the local equation of $Z(\mathbf{b}) \cap \widehat{\Omega}_{\Lambda}^{ord}$ in $\widehat{\Omega}_{\Lambda}^{ord} \simeq SpfW[T, (T^p - T)^{-1}]^{\vee}$, assuming the intersection is non-empty. Note that as $\Lambda = \Lambda^{\sharp}$ is a lattice of type 0, then by Remark 2.9, we must have $ord_p q^+(\mathbf{b}) > 0$; otherwise, $Z(\mathbf{b})$ would meet the special fibre only at a type-2 lattice.

Set

$$\mathbf{b}' := \Pi_{\mathbb{X}}^{-1} \circ \mathbf{b} \in \mathbb{V}_{\phi}^{-},$$

so that $ord_p q^-(\mathbf{b}') = ord_p q^+(\mathbf{b}) - 1 \geq 0$, and note

$$\begin{aligned} \varphi^{-}\mathbf{b}' &= \mathbf{b}'(f_1) = \Pi_{\mathbb{X}}^{-1}\mathbf{b}(f_1) = V\Pi_{\mathbb{X}}^{-1}\mathbf{b}(f_0) \\ &= \epsilon(b) \\ &= p^r(\alpha'_0 v_0 + \alpha'_1 v_1), \end{aligned}$$

since by assumption the basis $\{v_0, v_1\}$ was taken to be ϵ -invariant.

Furthermore, we have evident inclusions

$$Z(\mathbf{b}') \subset Z(\mathbf{b}) \subset Z(p \cdot \mathbf{b}').$$

If we let $I \subset W[T, (T^p - T)^{-1}]^{\vee}$ denote the ideal defining $Z(\mathbf{b}) \cap \widehat{\Omega}_{\Lambda}^{ord}$, we then have

$$(p^{r+1}(\alpha'_0 T + \alpha'_1)) \subset I \subset (p^r(\alpha'_0 T + \alpha'_1)),$$

by applying the result from Case 1 to the two antilinear homomorphisms \mathbf{b}' and $p \cdot \mathbf{b}'$.

Recall that we want to prove that I is generated by $f := p^{r+1}(\alpha'_0 T + \alpha'_1)$. Thus, it suffices to prove that $I \subset (p^{r+1}(\alpha'_0 T + \alpha'_1))$.

To this end, suppose $g \in I$. We write

$$g = p^r(\alpha'_0 T + \alpha'_1)g_0,$$

and we need to show that p divides g_0 .

Suppose not. Then the reduction modulo p of $(\alpha'_0 T + \alpha'_1)g_0$ is a non-zero rational function over \mathbb{F} and so there exists $t \neq 0 \in \mathbb{F}$ such that

$$(\alpha'_0 t + \alpha'_1)g_0(t) \neq 0 \in \mathbb{F}.$$

Choose any preimage $\tilde{t} \in W_{r+1}^{\times}$ of t . Then the map

$$W[T, (T^p - T)^{-1}]^{\vee} \rightarrow W_{r+1}, \quad T \mapsto \tilde{t}$$

does *not* factor through $W[T, (T^p - T)^{-1}]^{\vee}/I$, which contradicts the assertion (2.29). Hence, we have that p divides g_0 , and so

$$I = (p^{r+1}(\alpha'_0 T + \alpha'_1)) = (p(a'_0 T + a'_1))$$

as required. □

For a type 2 lattice Λ' , we describe the analogous result: choose an ϵ -invariant basis $\{w_0, w_1\}$ for Λ' such that $q(w_0) = q(w_1) = 0$ and $h(w_0, w_1) = p^{-1}\delta$. Suppose $\mathbf{b} \in \mathbb{V}_{\phi}^{\pm}$, and let $b = \varphi^{\pm}\mathbf{b}$ denote the corresponding vector. If $Z(\mathbf{b}) \cap \widehat{\Omega}_{\Lambda'}^{ord} \neq \emptyset$, then by Proposition 2.7, we must have $b \in \Lambda'$, and so we may write

$$b = a_0 w_0 + a_1 w_1, \quad a_0, a_1 \in o_{k,p}.$$

The proof of the following proposition is completely analogous to Proposition 2.10, and is therefore omitted.

Proposition 2.11. *Let the notation be as in the previous paragraph. Then if $Z(\mathbf{b}) \cap \widehat{\Omega}_{\Lambda'}^{ord} \neq \emptyset$, we have*

$$Z(\mathbf{b}) \cap \widehat{\Omega}_{\Lambda'}^{ord} \simeq \text{Spf } W[T, (T^p - T)^{-1}]^\vee / (f),$$

where

$$f = \begin{cases} a_0 + a_1 T & \text{if } \mathbf{b} \in \mathbb{V}_\phi^+ \\ a'_0 + a'_1 T & \text{if } \mathbf{b} \in \mathbb{V}_\phi^- . \end{cases}$$

□

Lemma 2.12. *For $b \in C$, write*

$$\text{ord}_p(q(b)) = \begin{cases} 2t, & \text{if } \text{ord}_p(q(b)) \text{ is even} \\ 2t - 1, & \text{otherwise.} \end{cases}$$

For any vertex lattice Λ , let

$$(2.30) \quad r(b, \Lambda) := \max\{r \in \mathbb{Z} \mid p^{-r}b \in \Lambda\};$$

note that here we do not assume $b \in \Lambda$, and so $r(b, \Lambda)$ may be negative. Finally, let Λ_\odot denote the unique vertex lattice containing $p^{-t}b$, as in Lemma 2.8. Recall that Λ_\odot is type 0 (resp. type 2) if $\text{ord}_p q(b)$ is even (resp. odd).

Then we have the formula

$$r(b, \Lambda) = \begin{cases} t - \lfloor \frac{d(\Lambda, \Lambda_\odot)}{2} \rfloor, & \text{ord}_p q(b) \text{ even} \\ t - \lfloor \frac{d(\Lambda, \Lambda_\odot) + 1}{2} \rfloor, & \text{ord}_p q(b) \text{ odd.} \end{cases}$$

Here $d(\Lambda, \Lambda_\odot)$ is the distance function on \mathcal{B} , the Bruhat-Tits tree.

Proof. By scaling by a power of p , it suffices to prove this lemma in the case $t = 0$; that is, we may assume that either $\text{ord}_p q(b) = 0$ or $\text{ord}_p q(b) = -1$.

We proceed by induction on $d = d(\Lambda, \Lambda_\odot)$. If $d(\Lambda, \Lambda_\odot) = 0$, i.e. $\Lambda = \Lambda_\odot$, then $r(b, \Lambda_\odot) = 0$ by the definition of Λ_\odot .

Next, suppose we have proven the claim for all lattices L with $0 < d(L, \Lambda_\odot) \leq d$, and let Λ be a lattice with $d(\Lambda, \Lambda_\odot) = d$. We shall prove the desired formula holds for all the neighbours of Λ . There are four cases here to check, as Λ can be either type 0 or 2, and $\text{ord}_p q(b)$ can be even or odd.

For example, suppose that $\Lambda = \Lambda^\#$ and $\text{ord}_p q(b) = 0$. Then $d = d(\Lambda, \Lambda_\odot)$ is even.

There exists an $\mathcal{O}_{k,p}$ basis $\{v_0, v_1\}$ for Λ with $(v_0, v_0) = (v_1, v_1) = 0$, and $(v_0, v_1) = -(v_1, v_0) = \delta$. With respect to this basis, a complete list of the $p + 1$ neighbours of Λ in the Bruhat-Tits tree are described as follows:

$$\begin{aligned} \Lambda'_\infty &:= \text{span}_{\mathcal{O}_{k,p}} \{p^{-1}v_0, v_1\} \\ \Lambda'_\alpha &:= \text{span}_{\mathcal{O}_{k,p}} \{v_0, p^{-1}\alpha v_0 + p^{-1}v_1\} \end{aligned}$$

as $\alpha \in \mathbb{Z}_p$ ranges over a complete set of representatives for $\mathbb{F}_p = \mathbb{Z}_p / p\mathbb{Z}_p$.

Without loss of generality, we may assume that $d(\Lambda'_\infty, \Lambda_\odot) = d - 1$, and so for all the other neighbours, $d(\Lambda'_\alpha, \Lambda_\odot) = d + 1$. We may write

$$\begin{aligned} b &= p^r(a_0 v_0 + a_1 v_1) = p^r(p a_0 \cdot (p^{-1}v_0) + a_1 \cdot v_1) \\ &= p^{r+1}(a_0 \cdot (p^{-1}v_0) + p^{-1}a_1 \cdot v_1) \end{aligned}$$

where $r = r(b, \Lambda) = -\lfloor d/2 \rfloor$.

Now, noting that d is even, the induction hypothesis applied to Λ and Λ'_∞ yields

$$r(b, \Lambda'_\infty) = -\lfloor (d-1)/2 \rfloor = -\lfloor d/2 \rfloor + 1 = r(b, \Lambda) + 1 = r + 1,$$

and so we must have $a_0 \in o_{k,p}^\times$ and $p|a_1$. Hence, by inspecting the remaining neighbours Λ'_α of Λ , we immediately see

$$r(b, \Lambda'_\alpha) = r = -\left\lfloor \frac{d+1}{2} \right\rfloor,$$

as required. The remaining cases all follow in the same manner. \square

We now determine the equations of the special cycles $Z(\mathbf{b})$ in the local ring at a superspecial point $x \in \mathbb{P}_\Lambda(\mathbb{F}) \cap \mathbb{P}_{\Lambda'}(\mathbb{F})$, for a pair of neighbouring vertex lattices Λ and Λ' of type 0 and 2 respectively. Recall that we have a (formal) affine open neighbourhood of x

$$\widehat{\Omega}_{[\Lambda, \Lambda']} \simeq \text{Spf} \left(W[T_0, T_1, (T_0^{p-1} - 1)^{-1}, (T_1^{p-1} - 1)^{-1}] / (T_0 T_1 - p) \right)^\vee,$$

where the point x corresponds to the maximal ideal $\mathfrak{m}_x = (T_0, T_1)$.

Without loss of generality, we suppose that there is a basis $\{v_0, v_1\}$ for Λ , such that $(v_0, v_1) = -(v_1, v_0) = \delta$, $(v_0, v_0) = (v_1, v_1) = 0$, and

$$(2.31) \quad \Lambda = \text{span}\{v_0, v_1\}, \quad \Lambda' = \text{span}\{p^{-1}v_0, v_1\}.$$

Proposition 2.13. *Suppose $\mathbf{b} \in \mathbb{V}_\phi^\pm$, with $b = \varphi^\pm \mathbf{b} \in C$ the corresponding vector. Suppose $x \in Z(\mathbf{b})(\mathbb{F})$ is a superspecial point, with $x \in \mathbb{P}_\Lambda(\mathbb{F}) \cap \mathbb{P}_{\Lambda'}(\mathbb{F})$ as above. If we set*

$$r = r(b, \Lambda), \quad r' = r(b, \Lambda'),$$

as in (2.30), then the equation for the cycle $Z(\mathbf{b})$ in the local ring $\mathcal{O}_{\mathcal{D}, x}$ is given by

$$\begin{cases} (T_0)^{r'} (T_1)^r = 0, & \text{if } \mathbf{b} \in \mathbb{V}_\phi^- \\ (T_0)^{r'} (T_1)^{r+1} = 0, & \text{if } \mathbf{b} \in \mathbb{V}_\phi^+. \end{cases}$$

Proof. We begin with the case $\mathbf{b} \in \mathbb{V}_\phi^-$. With respect to a basis $\{v_0, v_1\}$ of Λ as in (2.31), we may write

$$(2.32) \quad \begin{aligned} b &= p^r (\alpha_0 v_0 + \alpha_1 v_1) \\ &= p^r (p \alpha_0 \cdot (p^{-1} v_0) + \alpha_1 v_1) \end{aligned}$$

where, as usual, $r = r(b, \Lambda)$, and at least one of α_0 or α_1 is a unit in $o_{k,p}$. Then in light of the choice of basis (2.31), we have

$$r' = \begin{cases} r, & \text{if } \alpha_1 \in o_{k,p}^\times \\ r+1, & \text{if } p \text{ divides } \alpha_1. \end{cases}$$

By the proof of Proposition 2.10, Case (i), we have that the special cycle $Z(\mathbf{b})$ is defined at x by the vanishing of the element

$$f := p^r (\alpha_0 T_0 + \alpha_1) \in \mathcal{O}_{\mathcal{D}, x}.$$

Now consider the case $r' = r$ (i.e. $\alpha_1 \in o_{k,p}^\times$). Then the factor $\alpha_0 T_0 + \alpha_1$ is a unit in $\mathcal{O}_{\mathcal{D}, x}$ and so $Z(\mathbf{b})$ is given by

$$p^r = (T_0)^r \cdot (T_1)^r = 0,$$

as required. If on the other hand, $r' = r + 1$, then p divides α_1 and $\alpha_0 \in \mathcal{O}_{k,p}^\times$. Thus

$$p^r(\alpha_0 T_0 + \alpha_1) = p^r T_0 (\alpha_0 + (\alpha_1/p) T_1) = p^r T_0 \cdot u$$

with $u = (\alpha_0 + (\alpha_1/p) T_1) \in \mathcal{O}_{\mathcal{D},x}^\times$, and so the cycle $Z(\mathbf{b})$ is given by

$$(T_0)^{r+1} (T_1)^r = (T_0)^{r'} (T_1)^r = 0,$$

as required.

The proof for a homomorphism $\mathbf{b} \in \mathbb{V}_\phi^+$ is completely analogous. \square

Theorem 2.14. *Suppose $\mathbf{b} \in \mathbb{V}_\phi^\pm$, with $\text{ord}_p q^\pm(\mathbf{b}) \geq 0$ and write*

$$\text{ord}_p q^\pm(\mathbf{b}) = \begin{cases} 2t, & \text{if } \text{ord}_p q^\pm(\mathbf{b}) \text{ is even,} \\ 2t - 1, & \text{if } \text{ord}_p q^\pm(\mathbf{b}) \text{ is odd.} \end{cases}$$

Let $b := \varphi^\pm \mathbf{b} \in C$ be the corresponding vector, and write $b = p^k b_\odot$, where $\text{ord}_p q(b_\odot)$ is either 0 or -1 . Then by Lemma 2.8, there is a unique vertex lattice Λ_\odot (the “central lattice”) such that $b_\odot \in \Lambda_\odot - p\Lambda_\odot$.

Finally, for any vertex lattice Λ we define

$$(2.33) \quad m(\mathbf{b}, \Lambda) := \begin{cases} 0, & \text{if } b \notin \Lambda \\ t - \lfloor d(\Lambda, \Lambda_\odot)/2 \rfloor, & \text{if } b \in \Lambda \text{ and } \text{ord}_p q^\pm(\mathbf{b}) \text{ is even,} \\ t - \lfloor (d(\Lambda, \Lambda_\odot) + 1)/2 \rfloor, & \text{if } b \in \Lambda \text{ and } \text{ord}_p q^\pm(\mathbf{b}) \text{ is odd.} \end{cases}$$

Then we have the equality of cycles on \mathcal{D} :

$$Z(\mathbf{b}) = Z(\mathbf{b})^{\text{hor}} + \sum_{\Lambda} m(\mathbf{b}, \Lambda) \mathbb{P}_\Lambda,$$

where $Z(\mathbf{b})^{\text{hor}} \simeq \text{Spf } W$ is a horizontal divisor meeting the special fibre of \mathcal{D} at a single ordinary special point in the component $\mathbb{P}_{\Lambda_\odot}$.

Proof. To start, we have the following relations:

$$m(\mathbf{b}, \Lambda) = \begin{cases} 0, & \text{if } b \notin \Lambda \\ r(b, \Lambda) + 1, & \text{if } b \in \Lambda, \Lambda^\# = \Lambda \\ r(b, \Lambda), & \text{if } b \in \Lambda, \Lambda^\# = p\Lambda \end{cases} \quad \text{for } \mathbf{b} \in \mathbb{V}_\phi^+, b = \varphi^+ \mathbf{b}$$

and

$$m(\mathbf{b}, \Lambda) = \begin{cases} 0, & \text{if } b \notin \Lambda \\ r(b, \Lambda), & \text{if } b \in \Lambda \end{cases} \quad \text{for } \mathbf{b} \in \mathbb{V}_\phi^-, b = \varphi^- \mathbf{b}$$

which are easily verified by comparing the definition of $m(\mathbf{b}, \Lambda)$ above with Lemma 2.12.

The proof of this theorem amounts to collating the information contained in the local equations given by Propositions 2.10, 2.11, and 2.13. We shall illustrate this argument in the case $\mathbf{b} \in \mathbb{V}_\phi^-$ with $\text{ord}_p q^-(\mathbf{b}) = 2t$ even; the remaining cases follow in an identical manner.

Suppose $b = \varphi^- \mathbf{b} \in C$ is the vector corresponding to $\mathbf{b} \in \mathbb{V}_\phi^-$, so by assumption $\text{ord}_p q(b) = \text{ord}_p q^-(\mathbf{b}) = 2t$ is also even. In particular, we have $b_\odot = p^{-t} b$.

Let Λ denote a type 0 lattice, and choose a basis $\{v_0, v_1\}$ such that $h(v_0, v_1) = \delta$ and $q(v_0) = q(v_1) = 0$. If we write

$$b = p^r(\alpha_0 v_0 + \alpha_1 v_1) = p^r b_0,$$

where $r = r(b, \Lambda)$ and $b_0 \in \Lambda - p\Lambda$ as usual, then by Proposition 2.10, we have that the intersection $Z(\mathbf{b}) \cap \widehat{\Omega}_\Lambda^{ord}$ with the *ordinary locus* $\widehat{\Omega}_\Lambda^{ord}$ of $\widehat{\Omega}_\Lambda$ is defined by the vanishing of the element

$$(2.34) \quad p^r(\alpha_0 T + \alpha_1) \in W[T, (T^p - T)^{-1}]^\vee.$$

Recall that the superscript $^\vee$ indicates that we are taking the completion along the ideal generated by p .

Note that by Lemma 2.12, we have the following equivalencies:

$$ord_p q(b_0) = 0 \iff b_0 = b_\odot \iff r = r(b, \Lambda) = t \iff \Lambda = \Lambda_\odot$$

By the choice of basis $\{v_0, v_1\}$, we also have

$$q(b_0) = \delta(\alpha_0 \alpha'_1 - \alpha'_0 \alpha_1).$$

If $\Lambda \neq \Lambda_\odot$, so that $ord_p q(b_0) > 0$, then $\alpha_0 \alpha'_1 \equiv \alpha'_0 \alpha_1 \pmod{p}$. A short calculation implies that the term $(\alpha_0 T + \alpha_1)$ is a unit in $W[T, (T^p - T)^{-1}]^\vee$. Hence, for $\Lambda \neq \Lambda_\odot$, we have that $Z(\mathbf{b}) \cap \widehat{\Omega}_\Lambda^{ord}$ is determined by the equation

$$p^r = 0.$$

As the component \mathbb{P}_Λ^{ord} is given by the equation $p = 0$, we then obtain the equality of cycles

$$(2.35) \quad Z(\mathbf{b}) \cap \widehat{\Omega}_\Lambda^{ord} = m(\mathbf{b}, \Lambda) \mathbb{P}_\Lambda^{ord}, \quad \text{for } \Lambda \text{ of type 0, } \Lambda \neq \Lambda_\odot.$$

On the other hand, if $\Lambda = \Lambda_\odot$, we have $\alpha_0, \alpha_1 \in o_{k,p}^\times$, and so the cycle defined by (2.34) is

$$(2.36) \quad Z(\mathbf{b}) \cap \widehat{\Omega}_{\Lambda_\odot}^{ord} = Z(\mathbf{b})^{hor} + m(\mathbf{b}, \Lambda_\odot) \mathbb{P}_{\Lambda_\odot}^{ord}$$

where

$$Z(\mathbf{b})^{hor} := \text{Spf } W[T, (T^p - T)^{-1}]^\vee / (\alpha_0 T + \alpha_1) \simeq \text{Spf } W.$$

Now suppose Λ' is a type 2 lattice. We choose a basis $\{w_0, w_1\}$ of Λ' such that $h(w_0, w_1) = p^{-1}\delta$ and $q(w_0) = q(w_1) = 0$. Writing

$$b = p^{r'}(\alpha_0 w_0 + \alpha_1 w_1),$$

with $r' = r(b, \Lambda')$ and $b_0 := \alpha_0 w_0 + \alpha_1 w_1$, we have

$$q(b_0) = p^{-1}\delta(\alpha_0 \alpha'_1 - \alpha'_0 \alpha_1).$$

By Proposition 2.11, the cycle $Z(\mathbf{b}) \cap \widehat{\Omega}_{\Lambda'}^{ord}$ is given by the vanishing of the element

$$(2.37) \quad p^{r'}(\alpha'_0 + \alpha'_1 T) \in W[T, (T^p - T)^{-1}]^\vee.$$

Since we have assumed at the outset that $ord_p q(b)$ is even, it follows that $ord_p q(b_0) > -1$. As before, this implies that the factor $(\alpha'_0 + \alpha'_1 T)$ is a unit, and so the cycle $Z(\mathbf{b}) \cap \widehat{\Omega}_{\Lambda'}^{ord}$ is given by the equation $p^{r'} = 0$. Hence, we have the equality of cycles

$$(2.38) \quad Z(\mathbf{b}) \cap \widehat{\Omega}_{\Lambda'}^{ord} = r' \mathbb{P}_{\Lambda'}^{ord} = m(\mathbf{b}, \Lambda') \mathbb{P}_{\Lambda'}^{ord}, \quad \text{for } \Lambda' \text{ type 2.}$$

Combining (2.35), (2.36) and (2.38), we have

$$Z(\mathbf{b})^{ord} = Z(\mathbf{b})^{hor} + \sum_{\Lambda} m(\mathbf{b}, \Lambda) \mathbb{P}_\Lambda^{ord},$$

where $Z(\mathbf{b})^{ord}$ denotes the restriction of $Z(\mathbf{b})$ to the ordinary locus of \mathcal{D} (i.e. the open formal subscheme formed by the complement of the superspecial points).

Now suppose x is a superspecial point lying in the intersection $\mathbb{P}_\Lambda \cap \mathbb{P}_{\Lambda'}$, for a type 0 lattice Λ and its type 2 neighbour Λ' . Then Proposition 2.13 tells us that in a neighbourhood of x , the special cycle $Z(\mathbf{b})$ is determined by the equation

$$(T_0)^{r'} \cdot (T_1)^r = 0.$$

Recall that in such a neighbourhood, the components \mathbb{P}_Λ and $\mathbb{P}_{\Lambda'}$ are given by the equations $T_1 = 0$ and $T_0 = 0$ respectively, and so meet $Z(\mathbf{b})$ with multiplicities $r = m(\mathbf{b}, \Lambda)$ and $r' = m(\mathbf{b}, \Lambda')$ respectively. Therefore, we have

$$Z(\mathbf{b}) = Z(\mathbf{b})^{hor} + \sum_{\Lambda} m(\mathbf{b}, \Lambda) \mathbb{P}_\Lambda$$

as required.

We have proven the theorem in the case of an anti-linear special homomorphism $\mathbf{b} \in \mathbb{V}_\phi^-$ with $ord_p q^-(\mathbf{b})$ even; the other cases all follow in a completely analogous manner. \square

2.3. Relation to local orthogonal special cycles. In this section, we briefly recall definition of the *local orthogonal special cycles* constructed by Kudla and Rapoport [KR1], and relate them to the unitary special cycles of the previous section.

Definition 2.15. *Suppose*

$$j \in \text{End}_{\mathcal{O}_{B,p}}(\mathbb{X})_{\mathbb{Q}_p} := \text{End}_{\mathcal{O}_{B,p}}(\mathbb{X}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

is an $\mathcal{O}_{B,p}$ -linear quasi-endomorphism of \mathbb{X} . Define $Z^o(j)$ to be the closed formal subscheme of \mathcal{D} which represents the following moduli problem: if $S \in \mathbf{Nilp}$, we take $Z^o(j)(S)$ to be the locus of points $(X, \iota_X, \rho_X) \in \mathcal{D}(S)$ such that the quasi-morphism

$$\rho_X^{-1} \circ j \circ \rho_X : X \times \overline{S} \rightarrow X \times \overline{S}$$

lifts to an endomorphism of X .

We also define $Z^o(j)^{pure}$ to be the Cohen-Macaulayfication of $Z^o(j)$, namely the closed subscheme of $Z^o(j)$ defined by the ideal sheaf of sections with finite support, cf. [KR1, §4].

These cycles (and their global counterparts, discussed in the next section) have been studied extensively by Kudla-Rapoport [KR1] and Kudla-Rapoport-Yang [KRY]; of immediate interest to us is the following special case of a result of Kudla and Rapoport, which is the counterpart to Theorem 2.14.

Proposition 2.16 ([KR1] Proposition 4.5). *Suppose $j \in \text{End}_{\mathcal{O}_{B,p}}(\mathbb{X})_{\mathbb{Q}_p}$ such that $j^2 = u^2 p^{2\alpha} \delta^2$ for some $u \in \mathbb{Z}_p^\times$ (so in particular, $\text{Tr}(j) = 0$, and $\mathbb{Q}_p(j)$ is isomorphic to k_p). Then $Z^o(j)^{pure}$ is a divisor on \mathcal{D} .*

Moreover, let Λ_0 be the unique vertex lattice such that $j(\Lambda_0) = p^\alpha \Lambda_0$. For any vertex lattice Λ , define

$$\mu^o(j, \Lambda) := \max \{ \alpha - d(\Lambda, \Lambda_0), 0 \}.$$

Then there is an equality of cycles on \mathcal{D} :

$$Z^o(j)^{pure} = Z^o(j)^{hor} + \sum_{\Lambda} \mu^o(j, \Lambda) \mathbb{P}_\Lambda,$$

where $Z^o(j)^{hor}$ is a disjoint sum of two divisors, each of which is isomorphic to $\text{Spf} W$ and meets the special fibre of \mathcal{D} at an ordinary special point in the component \mathbb{P}_{Λ_0} . \square

Suppose $\gamma \in \text{End}_{\mathcal{O}_{B,p}}(\mathbb{X})_{\mathbb{Q}_p}$. Then γ induces an endomorphism of $N(\mathbb{X})_0$, the 0-graded component of the rational Dieudonné module of \mathbb{X} , and commutes with the operators F and V . Therefore γ defines an $o_{k,p}$ -linear endomorphism on $C := N(\mathbb{X})_0^{V^{-1}F}$, which, abusing notation, we also denote by γ .

Now we observe that if j is as in Proposition 2.16, then as an endomorphism of C , its characteristic polynomial splits. In particular, we may find an eigenvector $b_0 \in C$ with eigenvalue $p^\alpha \delta$. By scaling, we may assume $\text{ord}_p q(b_0)$ is either 0 or -1 . Note that j also commutes with the Galois-semilinear operator $\epsilon := V^{-1} \circ \Pi_{\mathbb{X}}$, and so

$$(j \circ \epsilon)(b_0) = (\epsilon \circ j)(b_0) = -p^\alpha \delta \cdot \epsilon(b_0).$$

In other words, b_0 and $\epsilon(b_0)$ are eigenvectors for j , with eigenvalues $p^\alpha \delta$ and $-p^\alpha \delta$ respectively. Moreover, the “central” lattice Λ_0 attached to j , as in Proposition 2.16 (ii), is $\Lambda_0 = \text{span}\{b_0, \epsilon(b_0)\}$.

In anticipation of the next section, we make the following definition: for j as above, set

$$(2.39) \quad \nu_p(j) = \nu_p(j, \phi) := \begin{cases} 1, & \text{if there exists an eigenvector } b \in C \text{ with } \text{ord}_p q(b) \text{ odd} \\ p, & \text{otherwise.} \end{cases}$$

In other words, we have $\nu_p(j) = 1$ if and only if there exists a homomorphism $\mathbf{b} \in \mathbb{V}_\phi^+$ such that

$$j \circ \mathbf{b} = p^\alpha \cdot (\mathbf{b} \circ i_{\mathbb{Y}}(\delta)), \quad \text{with } \text{ord}_p q^+(\mathbf{b}) \text{ even.}$$

The following two theorems relate the orthogonal and unitary special cycles on \mathcal{D} .

Theorem 2.17. *Suppose $j \in \text{End}_{\mathcal{O}_{B,p}}(\mathbb{X})_{\mathbb{Q}_p}$, with $j^2 = u^2 p^{2\alpha} \delta^2$ for some $\alpha > 0$ and $u \in \mathbb{Z}_p^\times$. Abbreviate $\nu = \nu_p(j)$, and fix $b_0 \in C$ an eigenvector with eigenvalue $p^\alpha \delta$ such that $q(b_0) = p^{-1}\nu$.*

(i) *If α is even, define $\mathbf{b}^+ \in \mathbb{V}_\phi^+$ and $\mathbf{b}^- \in \mathbb{V}_\phi^-$ by the relations:*

$$\varphi^+(\mathbf{b}^+) = \nu^{-1} \cdot p^{\alpha/2} \cdot b_0, \quad \varphi^-(\mathbf{b}^-) = p^{\alpha/2} \cdot b_0.$$

(ii) *If α is odd, define \mathbf{b}^+ and \mathbf{b}^- by the relations:*

$$\varphi^+(\mathbf{b}^+) = p^{\frac{\alpha-1}{2}} \cdot b_0, \quad \varphi^-(\mathbf{b}^-) = \nu^{-1} \cdot p^{\frac{\alpha+1}{2}} \cdot b_0$$

Then we have an equality of cycles on \mathcal{D}

$$(2.40) \quad Z^o(j)^{\text{pure}} = Z(\mathbf{b}^+) + Z(\mathbf{b}^-).$$

Remark 2.18. The key feature of this formula is that in every case, exactly one of the special homomorphism \mathbf{b}^\pm appearing on the right hand side has norm p^α , while the other has norm $p^{\alpha-1}$ (up to units in \mathbb{Z}_p^\times). \diamond

Proof. Note that the central lattices for the cycles $Z^o(j)$, $Z(\mathbf{b}^+)$, and $Z(\mathbf{b}^-)$ are all the same lattice, namely $\Lambda_\odot := \text{span}_{o_{k,p}}\{b, \epsilon(b)\}$. One can easily verify that in every case in the statement of the theorem, we have

$$\begin{aligned} m(\mathbf{b}^+, \Lambda) + m(\mathbf{b}^-, \Lambda) &= \alpha - \lfloor d(\Lambda, \Lambda_\odot)/2 \rfloor - \lfloor (d(\Lambda, \Lambda_\odot) + 1)/2 \rfloor \\ &= \alpha - d(\Lambda, \Lambda_\odot) \\ &= \mu^o(j, \Lambda) \end{aligned}$$

for any vertex lattice Λ with $d(\Lambda, \Lambda_\odot) \leq \alpha$. Indeed, the vectors \mathbf{b}^+ and \mathbf{b}^- were scaled precisely so that this relation holds. On the other hand, if $d(\Lambda, \Lambda_\odot) > \alpha$, then neither side of (2.40) meets the component \mathbb{P}_Λ . Hence, \mathbb{P}_Λ always occurs with the same multiplicity on both sides of (2.40).

It therefore suffices to show the horizontal components of both sides, which live in the open affine $\widehat{\Omega}_{\Lambda_{\odot}}^{ord}$, are equal. Suppose Λ_{\odot} is type 0, i.e.

$$ord_p q(b_0) = \nu_p(j) = 0.$$

Let $\{v_0, v_1\}$ denote an ϵ -invariant basis for Λ_{\odot} , with $q(v_0) = q(v_1) = 0$, and $h(v_0, v_1) = \delta$, and write

$$b_0 = a_0 v_0 + a_1 v_1, \quad \epsilon(b_0) = a'_0 v_0 + a'_1 v_1.$$

Note that $q(b_0) = -q(\epsilon(b_0)) = \delta(a_0 a'_1 - a'_0 a_1)$ is a unit in \mathbb{Z}_p^\times , by assumption.

Let $j_0 := p^{-\alpha} j$. Then $(j_0)^2 = \delta^2$, and by [KR1, Proposition 3.2], we have

$$Z^o(j)^{hor} = Z^o(j_0).$$

As an endomorphism of C , the element j_0 is determined by the fact that b_0 and $\epsilon(b_0)$ are eigenvectors with eigenvalues δ and $-\delta$ respectively. It is therefore defined by the matrix

$$[j] = \frac{\delta}{a_0 a'_1 - a'_0 a_1} \begin{pmatrix} a_0 a'_1 + a'_0 a_1 & -2n(a_0) \\ 2n(a_1) & -a_0 a'_1 - a'_0 a_1 \end{pmatrix} \in M_2(\mathbb{Z}_p)$$

with respect to the basis $\{v_0, v_1\}$. Now in the affine neighbourhood

$$\widehat{\Omega}_{\Lambda_{\odot}}^{ord} \simeq Spf W[T, (T^p - T)^{-1}]^\vee,$$

the proof of Proposition 2.10 says that the cycle $Z(\mathbf{b}^+)^{hor} + Z(\mathbf{b}^-)^{hor}$ is given by the vanishing of

$$(2.41) \quad (a_0 T + a_1) \cdot (a'_0 T + a'_1) = n(a_0) T^2 + (a_0 a'_1 + a'_0 a_1) T + n(a_1).$$

On the other hand, by [KR1, Equation (3.5)], the equation for $Z^o(j_0)$ in $\widehat{\Omega}_{\Lambda_{\odot}}^{ord}$ is given by

$$\frac{-2\delta}{a_0 a'_1 - a'_0 a_1} (n(a_0) T^2 + (a_0 a'_1 + a'_0 a_1) T + n(a_1)),$$

which differs from (2.41) by a scalar in \mathbb{Z}_p^\times , and hence defines the same divisor.

If $\Lambda_{\odot} = \text{span}\{b_0, \epsilon(b_0)\}$ is a type 2 lattice, fix an ϵ -invariant basis $\{w_0, w_1\}$ for Λ_{\odot} , such that $h(w_0, w_1) = p^{-1}\delta$ and $q(w_0) = q(w_1) = 0$. As before, we write

$$b_0 = a_0 w_0 + a_1 w_1, \quad \epsilon(b_0) = a'_0 w_0 + a'_1 w_1,$$

and note that $a_0 a'_1 - a'_0 a_1 \in o_{k,p}^\times$.

The cycle $Z(\mathbf{b}^+)^{hor} + Z(\mathbf{b}^-)^{hor}$ in $\widehat{\Omega}_{\Lambda_{\odot}}^{ord}$ is then given by the vanishing of the element

$$(a_0 + a_1 T) \cdot (a'_0 + a'_1 T) = n(a_0) + (a_0 a'_1 + a'_0 a_1) T + n(a_1) T^2.$$

Turning to the special endomorphism j , we note that it acts on C via the matrix

$$[j] = \frac{\delta}{a_0 a'_1 - a'_0 a_1} \begin{pmatrix} a_0 a'_1 + a'_0 a_1 & -2n(a_0) \\ 2n(a_1) & -a_0 a'_1 - a'_0 a_1 \end{pmatrix}$$

with respect to the basis $\{w_0, w_1\}$. One can then check (e.g. by exploiting the action of $GL_2(\mathbb{Q}_p)$ on \mathcal{D} c.f. [KR1, §1]), that the equation for $Z^o(j_0)$ in this case is given by

$$\frac{-2\delta}{a_0 a'_1 - a'_0 a_1} (n(a_0) + (a_0 a'_1 + a'_0 a_1) T + n(a_1) T^2),$$

and hence defines the divisor $Z(\mathbf{b}^+)^{hor} + Z(\mathbf{b}^-)^{hor}$, as required. \square

We also have the corresponding theorem in the case $\alpha = 0$; note that all the cycles involved are horizontal.

Theorem 2.19. *Suppose $j \in \text{End}_{\mathcal{O}_{B,p}}(\mathbb{X})_{\mathbb{Q}_p}$ with $j^2 = u^2 \delta^2$ with $u \in \mathbb{Z}_p^\times$. Let $b_0 \in C$ denote an eigenvector with eigenvalue δ , and suppose that $q(b_0) = p^{-1} \nu_p(j, \phi)$, so $\text{ord}_p q(b_0)$ is 0 or -1 .*

Then if $\nu_p(j, \phi) = 1$, and we define $\mathbf{b}_1, \mathbf{b}_2 \in \mathbb{V}_\phi^+$ by the formulas

$$\mathbf{b}_1 = \varphi^+(b_0), \quad \mathbf{b}_2 = \varphi^+(\epsilon(b_0)),$$

then

$$(2.42) \quad Z^o(j)^{\text{pure}} = Z(\mathbf{b}_1) + Z(\mathbf{b}_2).$$

Similarly, if $\nu_p(j, \phi) = p$ and we define $\mathbf{b}_1, \mathbf{b}_2 \in \mathbb{V}_\phi^-$ by

$$\mathbf{b}_1 = \varphi^-(b_0), \quad \mathbf{b}_2 = \varphi^-(\epsilon(b_0)),$$

then

$$(2.43) \quad Z^o(j)^{\text{pure}} = Z(\mathbf{b}_1) + Z(\mathbf{b}_2).$$

Proof. The proof is completely analogous to the proof of Theorem 2.17, and is omitted. \square

3. GLOBAL SPECIAL CYCLES AND THETA SERIES

Notation: Let B be an indefinite division quaternion algebra over \mathbb{Q} of discriminant D_B , with the operation $b \mapsto b^t$ denoting the main involution, and Nrd and Trd the reduced norm and trace forms, respectively. We also fix a maximal order \mathcal{O}_B , and an element $\theta \in \mathcal{O}_B$ such that $\theta^t = -\theta$ and $\theta^2 = -D_B$.

Let $k = \mathbb{Q}(\sqrt{\Delta})$ denote an imaginary quadratic field, with $\Delta < 0$ a squarefree integer, and let o_k be the ring of integers. Let $a \mapsto a'$ denote the non-trivial Galois automorphism. For this section, the following assumptions will be in force: (i) we assume Δ is even, and (ii) every prime that divides D_B is *inert* in k .

The second condition above ensures that there exists an *optimal embedding* $\phi : o_k \rightarrow \mathcal{O}_B$; i.e. ϕ is an embedding of rings such that $\phi(o_k) = \phi(k) \cap \mathcal{O}_B$. Let Opt denote the set of all optimal embeddings, and let Opt/\mathcal{O}_B^\times denote the set of equivalence classes of optimal embeddings modulo the action of \mathcal{O}_B^\times given by

$$\epsilon \cdot \phi := Ad_{\epsilon^{-1}} \circ \phi, \quad \epsilon \in \mathcal{O}_B^\times.$$

We also fix an (odd) prime p that divides D_B , and so in particular is inert in k ; this prime should be viewed as the ‘same’ prime p in Section 2. Let $\mathbb{F} = \mathbb{F}_p^{\text{alg}}$ and $W = W(\mathbb{F})$ the ring of Witt vectors. We recall from the previous section that we have fixed an embedding $\tau_0 : o_k/(p) \rightarrow \mathbb{F}$, whose lift to characteristic zero is also denoted $\tau_0 : o_k \rightarrow W$. Additionally, we fix a trivialization of the prime-to- p roots of unity in \mathbb{F} :

$$\mathbb{A}_f^p(1) := \left(\prod_{\ell \neq p} \varprojlim_n \mu_{\ell^n}(\mathbb{F}) \right) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{A}_f^p$$

Finally, we set $\widehat{\mathbb{Z}}^p := \prod_{\ell \neq p} \mathbb{Z}_\ell$, and if M is any \mathbb{Z} -module, we define $\widehat{M}^p := M \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^p$.

3.1. Shimura curves and special cycles. In this section, we recall the construction of Shimura curves, and orthogonal and unitary special cycles on them.

Definition 3.1 (Shimura curve). *Let \mathcal{C}_B denote the moduli problem which associates to a scheme S over $\text{Spec } \mathbb{Z}$ the category whose objects are pairs*

$$\mathcal{C}_B(S) = \{(A, \iota)\},$$

where (i) A is an abelian surface over S , and (ii) $\iota : \mathcal{O}_B \rightarrow \text{End}_S(A)$ is an action of \mathcal{O}_B on A . We also require that for every $b \in \mathcal{O}_B$,

$$\det(T - \iota(b)|_{\text{Lie}(A)}) = T^2 - \text{Trd}(b)T + \text{Nrd}(b) \in \mathcal{O}_S[T].$$

We record some basic facts about \mathcal{C}_B :

Proposition 3.2 (cf. [KRY], Proposition 3.1.1). *The moduli problem \mathcal{C}_B is representable by a Deligne-Mumford (DM) stack, which we also denote by \mathcal{C}_B . It is regular, proper and flat over $\text{Spec } (\mathbb{Z})$ of relative dimension 1, and smooth over $\text{Spec } \mathbb{Z}[D_B^{-1}]$. \square*

The orthogonal special cycles, as constructed in e.g. [KRY], are also defined by a moduli problem:

Definition 3.3. *For $n \in \mathbb{Z}_{>0}$, let $\mathcal{Z}^o(n)$ denote the DM stack which represents the following moduli problem over $\text{Spec } \mathbb{Z}$: to a scheme S/\mathbb{Z} , we let $\mathcal{Z}^o(n)(S)$ be the category of tuples (A, ι, ξ) , where*

- (i) $(A, \iota) \in \mathcal{C}_B(S)$, and
- (ii) $\xi \in \text{End}_{\mathcal{O}_B}(A)$ is an \mathcal{O}_B -linear endomorphism of A with $\text{Tr}(\xi) = 0$ and $\xi^2 = -n$.

We view $\mathcal{Z}^o(n)$ as a cycle on \mathcal{C}_B via the natural forgetful map $\mathcal{Z}^o(n) \rightarrow \mathcal{C}_B$, which is finite and unramified by [KRY, §3.4]. Let $\mathcal{Z}^o(n)^{\text{pure}}$ denote the Cohen-Macaulayfication of $\mathcal{Z}^o(n)$, as in p. 55 of loc. cit., so that $\mathcal{Z}^o(n)^{\text{pure}}$ is of pure codimension 1 in \mathcal{C}_B .

We now turn to the unitary special cycles, following [KR3]. Fix once and for all an element $\theta \in \mathcal{O}_B$ such that $\theta^2 = -D_B$. Given a point $(A, \iota) \in \mathcal{C}_B(S)$, there exists a unique principal polarization λ_A^0 on A such that

$$(\lambda_A^0)^{-1} \circ \iota(b)^\vee \circ \lambda_A^0 = \iota(\theta^{-1} b^\vee \theta), \quad \text{for all } b \in \mathcal{O}_B,$$

cf. [How, §3.1]. For any optimal embedding $\phi : o_k \rightarrow \mathcal{O}_B$, we may then define a new (non-principal) polarization

$$(3.1) \quad \lambda_{A, \phi} := \lambda_A^0 \circ \iota(\theta \phi(\sqrt{\Delta})).$$

Note that the Rosati involution $*$ associated to this latter polarization satisfies the property

$$\iota_A(\phi(a))^* = \iota_A(\phi(a')).$$

for all $a \in o_k$.

Let \mathcal{E} denote the moduli stack over $\text{Spec } \mathbb{Z}$ whose S -points parametrize triples

$$\mathcal{E}(S) = \{\underline{E} = (E, i_E, \lambda_E)\};$$

here E/S is an elliptic curve, $i_E : o_k \rightarrow \text{End}_S(E)$ is an o_k -action, and λ_E is a principal polarization.

Suppose we are given two points $\underline{E} \in \mathcal{E}(S)$ and $\underline{A} \in \mathcal{C}_B(S)$, for some base scheme S . Then, for any optimal embedding $\phi : o_k \rightarrow \mathcal{O}_B$, we may define the space of o_k -linear homomorphisms:

$$\text{Hom}_{o_k, \phi}(\underline{E}, \underline{A}) := \{y \in \text{Hom}_S(E, A) \mid y \circ i_E(a) = \iota_A(\phi(a)) \circ y \text{ for all } a \in o_k\}.$$

We also define an o_k -hermitian form $h_{E,A}^\phi$ on this space by the formula

$$h_{E,A}^\phi(y_1, y_2) := \lambda_E^{-1} \circ y_2^\vee \circ \lambda_{A,\phi} \circ y_1 \in \text{End}_{o_k}(E) \simeq o_k.$$

Definition 3.4 (Unitary special cycles.). *Suppose $m \in \mathbb{Z}_{>0}$ and $\phi: o_k \rightarrow \mathcal{O}_B$ is an optimal embedding. Let $\mathcal{Z}(m, \phi)$ denote the DM stack over $\text{Spec } \mathbb{Z}$ representing the following moduli problem: for a scheme S/\mathbb{Z} , we define $\mathcal{Z}(m, \phi)(S)$ to be the category of tuples*

$$\mathcal{Z}(m, \phi)(S) = \left\{ (\underline{E}, \underline{A}, y) \right\}$$

where (i) $\underline{E} \in \mathcal{E}(S)$, (ii) $\underline{A} \in \mathcal{C}_B(S)$, and (iii) $y \in \text{Hom}_{o_k, \phi}(\underline{E}, \underline{A})$ such that $h_{E,A}^\phi(y, y) = m$.

The forgetful map $\mathcal{Z}(m, \phi) \rightarrow \mathcal{C}_B$ is finite and unramified by the proof of [KR3, Proposition 2.10], which applies verbatim to the present setting, and so we view $\mathcal{Z}(m, \phi)$ as a cycle on \mathcal{C}_B .

Remark 3.5. Let \mathcal{E}^+ denote the moduli stack over $\text{Spec}(o_k)$ whose S -points parametrize triples (E, i_E, λ_E) , such that the induced action

$$\text{Lie}(i_E): o_k \rightarrow \text{End}(\text{Lie}(E))$$

agrees with the action induced by the structural morphism $o_k \rightarrow \mathcal{O}_S$. Similarly, let \mathcal{E}^- denote the stack over $\text{Spec}(o_k)$ parametrizing triples (E, i_E, λ_E) where the action induced by i_E on $\text{Lie}(E)$ is the *conjugate* of that induced by the structural morphism. Then, upon base change to $\text{Spec } o_k[\Delta^{-1}]$, we obtain a disjoint decomposition

$$\mathcal{E} \times_{\mathbb{Z}} \text{Spec } o_k[\Delta^{-1}] = (\mathcal{E}^+ \times_{o_k} \text{Spec } o_k[\Delta^{-1}]) \coprod (\mathcal{E}^- \times_{o_k} \text{Spec } o_k[\Delta^{-1}]).$$

There is a corresponding decomposition

$$\mathcal{Z}(m, \phi) \times \text{Spec } o_k[\Delta^{-1}] = \mathcal{Z}^+(m, \phi) \coprod \mathcal{Z}^-(m, \phi)$$

of stacks over $\text{Spec } o_k[\Delta^{-1}]$, where $\mathcal{Z}^+(m, \phi)$ (resp. $\mathcal{Z}^-(m, \phi)$) is the stack whose S -points parametrize tuples $(\underline{E}, \underline{A}, y)$ as before, with the added condition $\underline{E} \in \mathcal{E}^+(S)$ (resp. $\underline{E} \in \mathcal{E}^-(S)$). \diamond

3.2. The Shimura lift and the main theorem. We begin by briefly recalling the Shimura lift [Shim], which is a classical operation that takes modular forms of half-integral weight to modular forms of even weight; we shall follow the reformulation due to Niwa and Shintani [Niwa, Shin] (see also [Cip]) which extends Shimura's original construction to forms that are not Hecke eigenforms.

Proposition 3.6 (Shimura, Niwa, Cipra). *Let $\kappa \geq 3$ be an odd integer, and suppose F is a modular form of weight $\kappa/2$ for the group $\Gamma_0(4N)$ with nebentcharacter χ , and Fourier expansion*

$$F(\tau) = \sum_{n \geq 0} a(n) q^n \quad q := e^{2\pi i \tau}.$$

For a square-free integer $t \in \mathbb{Z}_{>0}$, we define a new character

$$\chi_t(a) := \chi(a) \left(\frac{-1}{a} \right)^{(\kappa-1)/2} \left(\frac{t}{a} \right),$$

where (\cdot/\cdot) denotes Shimura's modification of the Kronecker symbol, cf. [Cip, Appendix A]. We define another collection $\{b(m)\}_{m \geq 0}$ as follows: for $m > 0$, we let

$$(3.2) \quad b(m) := \sum_{n|m} \chi_t(n) n^{(\kappa-3)/2} a\left(t \frac{m^2}{n^2}\right).$$

To define $b(0)$, consider the Gauss sum

$$\check{\chi}_t(a) := \sum_{h=0}^{4Nt-1} \chi_t(h) \exp(2\pi i a h / 4Nt).$$

We then set

$$(3.3) \quad b(0) := \frac{1}{4} \left(\frac{-2i}{\pi} \right)^\lambda (Nt)^{\lambda-1} \Gamma(\lambda) L(\lambda, \check{\chi}_t) \cdot a(0), \quad \text{where } \lambda = (\kappa - 1)/2.$$

Then there is a holomorphic modular form $Sh_t(F)$, called the “Shimura lift of F , with parameter t ” of weight 2λ and level $2Nt$ with character χ^2 , whose Fourier expansion at ∞ is

$$(3.4) \quad Sh_t(F)(\tau) = \sum_{m \geq 0} b(m) q^{tm}.$$

Proof. This is Proposition 2.17 of [Cip]. \square

Our next step is to recall how certain intersection numbers involving the orthogonal special cycles $\mathcal{Z}^o(n)$ are related to Fourier coefficients of modular forms of weight $3/2$, as in [KRY, §4.3].

Let \mathcal{Y} denote an irreducible component of the fibre $(\mathcal{C}_B)_p$ of the Shimura curve, where $p|D_B$ is a prime of bad reduction, and suppose that \mathcal{Z} is a closed subscheme of \mathcal{C}_B . We define the pairing

$$\langle \mathcal{Z}, \mathcal{Y} \rangle := \log(p) \cdot \chi(\mathcal{O}_{\mathcal{Y}} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}}),$$

where χ is the stack version of the Euler-Poincaré characteristic, which takes into account the automorphism groups of points, cf. [DeRa, §VI.4]. We then have the following:

Theorem 3.7 (Kudla-Rapoport-Yang). *The generating series*

$$(3.5) \quad \Phi_{\mathcal{Y}}^o(\tau) := -\langle \omega, \mathcal{Y} \rangle + \sum_{n > 0} \langle \mathcal{Z}^o(n), \mathcal{Y} \rangle q^n, \quad q = e^{2\pi i \tau}$$

is (the q -expansion of) a modular form of weight $3/2$ and level $4D_B$ with trivial character; here $\omega = \omega_{\mathcal{C}_B/\mathbb{Z}}$ is the relative dualizing sheaf of \mathcal{C}_B , and

$$\langle \omega, \mathcal{Y} \rangle := \log(p) \cdot \deg(\omega|_{\mathcal{Y}})$$

is the product of $\log(p)$ and the degree of the restriction of ω to \mathcal{Y} .

Proof. This is [KRY, Theorem 4.3.4]; see Remark 4.3.5 of loc. cit. for a discussion of the level. The theorem is proved by identifying this generating series with the Fourier expansion of the theta series associated to a positive-definite ternary quadratic form. \square

Let $\chi_k = (\Delta/\cdot)$ denote the quadratic character associated to $k = \mathbb{Q}(\sqrt{\Delta})$, so that for any prime ℓ , we have (recalling $|\Delta|$ is even)

$$(3.6) \quad \chi_k(\ell) = \begin{cases} 1, & \text{if } \ell \text{ is split in } k \\ 0, & \text{if } \ell \text{ is ramified in } k \\ -1, & \text{if } \ell \text{ is inert in } k. \end{cases}$$

Let χ'_k denote the corresponding character induced to level $4|\Delta|D_B$, so that

$$\chi'_k(a) := \begin{cases} 0, & (a, 4D_B) \neq 1 \\ \chi_k(a), & (a, 4D_B) = 1, \end{cases}$$

and define $\check{\chi}'_k$ to be the Gauss sum

$$\check{\chi}'_k(a) := \sum_{h \bmod 4|\Delta|D_B} \chi'_k(h) \exp(2\pi i a h / 4|\Delta|D_B).$$

Theorem 3.8 (Main theorem). *For any irreducible component \mathcal{Y} of $(\mathcal{C}_B)_p$, define the “unitary generating series”*

$$\begin{aligned} \Phi_{\mathcal{Y}}^u(\tau) &:= \frac{i}{2\pi} L(1, \check{\chi}'_k) \langle \omega, \mathcal{Y} \rangle \\ &+ \sum_{m>0} \left(\frac{1}{2h(k)} \sum_{[\phi] \in \mathcal{O}_{pt} / \mathcal{O}_B^\times} \left\langle \mathcal{Z}(m, \phi) + \mathcal{Z}\left(\frac{m}{\gcd(m, D_B)}, \phi\right), \mathcal{Y} \right\rangle \right) q^m, \end{aligned}$$

where $q = e^{2\pi i \tau}$. There there is an equality of q -expansions

$$Sh_{|\Delta|} \Phi_{\mathcal{Y}}^o(\tau) = \Phi_{\mathcal{Y}}^u(\tau).$$

Proof. We apply Proposition 3.6 to the modular form $\Phi_{\mathcal{Y}}^o$, for which the relevant parameters are $\kappa = 3$, $N = D_B$, and χ the trivial character modulo $4D_B$, and we consider the Shimura lift of parameter $t = |\Delta|$. Note that the character χ'_k is precisely the character denoted χ_t in Proposition 3.6, and so the constant terms of $Sh_{|\Delta|} \Phi_{\mathcal{Y}}^o$ and $\Phi_{\mathcal{Y}}^u$ are equal, cf. (3.3).

By (3.2), and keeping in mind the shift by $t = |\Delta|$ on the right hand side of (3.4), it therefore remains to check that for all $m > 0$, we have the following equalities: (i) if $|\Delta|$ does not divide m , then

$$(3.7) \quad \frac{1}{2h(k)} \sum_{[\phi]} \left\langle \mathcal{Z}(m, \phi) + \mathcal{Z}\left(\frac{m}{\gcd(D_B, m)}, \phi\right), \mathcal{Y} \right\rangle \stackrel{?}{=} 0,$$

and (ii) if $m = |\Delta|m'$, then

$$(3.8) \quad \begin{aligned} &\frac{1}{2h(k)} \sum_{[\phi]} \left\langle \mathcal{Z}(m, \phi) + \mathcal{Z}\left(\frac{m}{\gcd(D_B, m)}, \phi\right), \mathcal{Y} \right\rangle \\ &\stackrel{?}{=} \sum_{\substack{\alpha|m' \\ (\alpha, D_B)=1}} \chi_k(\alpha) \left\langle \mathcal{Z}^o\left(|\Delta| \frac{(m')^2}{\alpha^2}\right), \mathcal{Y} \right\rangle. \end{aligned}$$

As discussed in [KR1, §4], the intersection multiplicity remains unchanged if one replaces $\mathcal{Z}^o(n)$ with $\mathcal{Z}^o(n)^{pure}$. Let $\tilde{\mathcal{Z}}^o(n)^{pure}$ (resp. $\tilde{\mathcal{Z}}^\pm(m, \phi)$) denote the base-change to $W = W(\mathbb{F})$ of the formal completion of $\mathcal{Z}^o(n)^{pure}$ (resp. $\mathcal{Z}^\pm(m, \phi)$) along its fibre at p . By Remark 4.4 of loc. cit. we may also replace $\mathcal{Z}^o(n)^{pure}$ and $\mathcal{Z}^\pm(m, \phi)$ by $\tilde{\mathcal{Z}}^o(n)^{pure}$ and $\tilde{\mathcal{Z}}^\pm(m, \phi)$. The proof of the relations (3.7) and (3.8) then follow immediately from Theorem 3.21 below; the remainder of this paper is devoted to the proof of this latter result. \square

Remark 3.9. The constant term of $\Phi_{\mathcal{Y}}^u$ can be re-expressed as follows:

$$\frac{i}{2\pi} L(1, \check{\chi}') \langle \omega, \mathcal{Y} \rangle = -\frac{h(k)}{|o_k^\times|} \prod_{\ell|D_B} \left(\frac{2\ell^2 + \ell - 1}{\ell^2} \right) \langle \omega, \mathcal{Y} \rangle,$$

by evaluating the L -function on the left via Dirichlet’s class number formula.

3.3. p -adic uniformizations. If \mathcal{X} denotes any of the stacks we have defined so far, we let $\tilde{\mathcal{X}}$ denote the base change to $W = W(\mathbb{F}_p^{alg})$ of the formal completion of its fibre at p . The p -adic uniformizations we are about to describe relate these completions to the moduli spaces of p -divisible groups discussed in the previous section.

We begin with the Shimura curve \mathcal{C}_B . Fix a pair $\underline{\mathbf{A}} = (\mathbf{A}, \iota_{\mathbf{A}}) \in \mathcal{C}_B(\mathbb{F})$, such that the p -divisible group

$$\underline{\mathbf{A}}[p^\infty] = (\mathbf{A}[p^\infty], \iota_{\mathbf{A}} \otimes \mathbb{Z}_p) \in \mathcal{D}(\mathbb{F}),$$

together with the induced $\mathcal{O}_{B,p}$ -action, is equal to the “base-point” (\mathbb{X}, ι_X) that we had fixed in defining the Drinfeld moduli space \mathcal{D} , cf. Definition 2.1. Let $B' = \text{End}_{\mathcal{O}_B}(\mathbf{A}) \otimes_{\mathbb{Z}} \mathbb{Q}$. Then B' is a quaternion algebra over \mathbb{Q} whose invariants differ from those of B at exactly p and ∞ . We let $H' = (B')^\times$, viewed as an algebraic group over \mathbb{Q} , and define

$$H'(\mathbb{Q})^* := \{h' \in H'(\mathbb{Q}) \mid \text{ord}_p \text{Nrd}'(h') = 0\},$$

where Nrd' is the reduced norm on B' . Let $K^p \subset H'(\mathbb{A}_f^p)$ denote the image of $(\widehat{\mathcal{O}_B^p})^\times$ under the natural identification $B'(\mathbb{A}_f^p) \simeq B^{op}(\mathbb{A}_f^p)$, and set

$$\Gamma' := K^p \cap H'(\mathbb{Q})^*.$$

Then Γ' acts on the Drinfeld upper-half plane \mathcal{D} as follows: if $\underline{X} = (X, \iota_X, \rho_X) \in \mathcal{D}(S)$ for some $S \in \mathbf{Nilp}$, and $\gamma \in \Gamma'$, then we set

$$\gamma \cdot \underline{X} := (X, \iota_X, \gamma_p \circ \rho_X),$$

where γ_p denotes the image of γ in $B' \otimes_{\mathbb{Q}} \mathbb{Q}_p = \text{End}_{\mathcal{O}_{B,p}}(\mathbb{X}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. We then obtain the following p -adic uniformization of \mathcal{C}_B :

Theorem 3.10. *There is an isomorphism of formal stacks over \mathbf{Nilp} :*

$$(3.9) \quad \widetilde{\mathcal{C}_B} \simeq [\Gamma' \backslash \mathcal{D}].$$

Proof. This is a restatement of [BC, Theorem III.5.3], see also [KR1, §8]. Primarily to set up notation for the sequel, we indicate the idea of the proof. We begin by fixing an \mathcal{O}_B -linear isomorphism

$$\eta_{\mathbf{A}} : Ta^p(\mathbf{A}) \xrightarrow{\sim} \widehat{\mathcal{O}_B^p},$$

where $Ta^p(\mathbf{A}) := \prod_{\ell \neq p} Ta_\ell(\mathbf{A})$ is the prime-to- p Tate module, and $\widehat{\mathcal{O}_B^p} := \mathcal{O}_B \otimes \widehat{\mathbb{Z}^p}$, which is viewed as an \mathcal{O}_B -module via left-multiplication.

Let $(A, \iota) \in \mathcal{C}_B(S)$, for some $S \in \mathbf{Nilp}$. Then there exists an \mathcal{O}_B -linear quasi-isogeny

$$(3.10) \quad \psi_A : A \rightarrow \mathbf{A}$$

such that (i) the induced map on p -divisible groups (over $\overline{S} = S \times_W \mathbb{F}$)

$$\psi_A[p^\infty]_{\overline{S}} : A[p^\infty] \times \overline{S} \rightarrow \mathbf{A}[p^\infty] \times \overline{S} = \mathbb{X} \times \overline{S}$$

is a quasi-isogeny of height 0, and (ii) the composition

$$\eta_{\mathbf{A}} \circ \psi_A^p : Ta^p(A)_{\mathbb{Q}} \rightarrow B(\mathbb{A}_f^p)$$

maps $Ta^p(A)$ isomorphically onto $\widehat{\mathcal{O}_B^p}$. Noting that any two quasi-isogenies satisfying both these properties necessarily differ by an element of Γ' , it follows that the map

$$\widetilde{\mathcal{C}_B} \rightarrow [\Gamma' \backslash \mathcal{D}], \quad (A, \iota) \mapsto [A[p^\infty], \iota \otimes \mathbb{Z}_p, \psi_A[p^\infty]_{\overline{S}}]$$

is an isomorphism. □

We have a similar statement for the orthogonal special cycles (Definition 3.3).

Theorem 3.11. *Let*

$$(3.11) \quad \Omega^o(n) := \left\{ \xi \in B'(\mathbb{Q}) \mid \text{Tr} d'(\xi) = 0, \xi^2 = -n, \text{ and } \eta_{\mathbf{A}} \circ Ta^p(\xi) \circ \eta_{\mathbf{A}}^{-1} \in \text{End}(\widehat{\mathcal{O}}_{\mathbf{B}}^p) \right\},$$

and note that Γ' acts on $\Omega^o(n)$ by conjugation. Then there is an isomorphism

$$\widetilde{\mathcal{Z}}^o(n) \simeq \left[\Gamma' \setminus \coprod_{\xi \in \Omega^o(n)} Z^o(\xi[p^\infty]) \right]$$

as formal stacks over W .

Proof. This follows by rewriting the conclusion of [?] in terms of the uniformization given in Theorem 3.10. \square

Note that for the Cohen-Macaulayfication $\widetilde{\mathcal{Z}}^o(n)^{pure}$ of $\widetilde{\mathcal{Z}}^o(n)$ we have that

$$\widetilde{\mathcal{Z}}^o(n)^{pure} \simeq \left[\Gamma' \setminus \coprod_{\xi \in \Omega^o(n)} Z^o(\xi[p^\infty])^{pure} \right].$$

We now turn to the p -adic uniformization of the unitary special cycles, following [KR3, §6]. Recall that we had fixed an embedding $\tau_0 : o_k/(p) \rightarrow \mathbb{F} = \mathbb{F}_p^{alg}$, which lifts uniquely to an embedding $\tau_0 : o_k \rightarrow W = W(\mathbb{F})$. Via these maps, we may view both \mathbb{F} and W as $o_k[\Delta^{-1}]$ -algebras, and in particular, we obtain a decomposition as in Remark 3.5:

$$\widetilde{\mathcal{Z}}(m, \phi) = \widetilde{\mathcal{Z}}^+(m, \phi) \coprod \widetilde{\mathcal{Z}}^-(m, \phi).$$

Fix a triple $\underline{\mathbf{E}} = (\mathbf{E}, i_{\mathbf{E}}, \lambda_{\mathbf{E}}) \in \mathcal{E}^+(\mathbb{F})$. We may also identify the p -divisible group $\underline{\mathbf{E}}[p^\infty]$, with its extra data, with the triple $\underline{\mathbf{Y}} = (\mathbf{Y}, i_{\mathbf{Y}}, \lambda_{\mathbf{Y}})$ of the previous section. We may further assume, by replacing \mathbf{E} with an isogenous elliptic curve if necessary, that there is an o_k -linear isomorphism

$$(3.12) \quad \eta_{\mathbf{E}} : Ta^p(\mathbf{E}) \xrightarrow{\sim} \widehat{o}_k^p.$$

We may further assume that the pullback of the symplectic form

$$(a, b) \mapsto \text{tr}(ab'(\sqrt{\Delta})^{-1})$$

via $\eta_{\mathbf{E}}$ is equal to the Weil pairing

$$e_{\mathbf{E}} : Ta^p(\mathbf{E})_{\mathbb{Q}} \times Ta^p(\mathbf{E})_{\mathbb{Q}} \rightarrow \mathbb{A}_f^p(1) \simeq \mathbb{A}_f^p.$$

defined by $\lambda_{\mathbf{E}}$. The two base points $\underline{\mathbf{E}} = (\mathbf{E}, i_{\mathbf{E}}, \lambda_{\mathbf{E}})$ and $\underline{\mathbf{A}} = (\mathbf{A}, \iota_{\mathbf{A}})$ allow us to define the global analogues of the spaces of special homomorphisms of the previous section, as follows. For any optimal embedding $\phi : o_k \hookrightarrow \mathcal{O}_{\mathbf{B}}$, let

$$\mathcal{V}_{\phi}^+ := \{ \beta \in \text{Hom}(\mathbf{E}, \mathbf{A})_{\mathbb{Q}} \mid \beta \circ i_{\mathbf{E}}(a) = \iota_{\mathbf{A}}(\phi(a)) \circ \beta \text{ for all } a \in o_{k,p} \}$$

and

$$\mathcal{V}_{\phi}^- := \{ \beta \in \text{Hom}(\mathbf{E}, \mathbf{A})_{\mathbb{Q}} \mid \beta \circ i_{\mathbf{E}}(a) = \iota_{\mathbf{A}}(\phi(a')) \circ \beta \text{ for all } a \in o_{k,p} \}.$$

We view these spaces as k -vector spaces via the action

$$a \cdot \beta := \beta \circ i_{\mathbf{E}}(a), \quad \text{for all } a \in k, \beta \in \mathcal{V}_{\phi}^{\pm}.$$

Recall that for any optimal embedding ϕ , we had defined a (non-principal) polarization $\lambda_{\mathbf{A},\phi}$ on \mathbf{A} , as in (3.1). The spaces \mathcal{V}_ϕ^\pm can then be equipped with Hermitian forms h^\pm , defined by the formulas:

$$(3.13) \quad h^+(\beta_1, \beta_2) := \lambda_{\mathbf{E}}^{-1} \circ \beta_2^\vee \circ \lambda_{\mathbf{A},\phi} \circ \beta_1 \in \text{End}_{o_k}(\mathbf{E})_{\mathbb{Q}} \simeq k, \quad \beta_1, \beta_2 \in \mathcal{V}_\phi^+$$

and

$$(3.14) \quad h^-(\beta_1, \beta_2) := \lambda_{\mathbf{E}}^{-1} \circ \beta_1^\vee \circ \lambda_{\mathbf{A},\phi} \circ \beta_2 \in \text{End}_{o_k}(\mathbf{E})_{\mathbb{Q}} \simeq k, \quad \beta_1, \beta_2 \in \mathcal{V}_\phi^-$$

respectively. Let $q^\pm(\beta) = h^\pm(\beta, \beta)$ denote the associated quadratic forms. We may then state the p -adic uniformization of the unitary special cycles, as follows:

Theorem 3.12. *Suppose $m \in \mathbb{Z}_{>0}$. For any fractional ideal \mathfrak{a} in k , and any optimal embedding $\phi : o_k \rightarrow \mathcal{O}_B$, let*

$$(3.15) \quad \Omega^\pm(m, \mathfrak{a}, \phi) := \left\{ \beta \in \mathcal{V}_\phi^\pm \mid q^\pm(\beta) = m \cdot \frac{p^{\text{ord}_p N(\mathfrak{a})}}{N(\mathfrak{a})}, \quad \eta_{\mathbf{A}} \circ \beta^p \circ \eta_{\mathbf{E}}^{-1}(\widehat{\mathfrak{a}}^p) \subset \widehat{\mathcal{O}_B^p} \right\},$$

where, for any $\beta \in \mathcal{V}_\phi^\pm$, we denote by $\beta^p : Ta^p(\mathbf{E})_{\mathbb{Q}} \rightarrow Ta^p(\mathbf{A})_{\mathbb{Q}}$ the induced map on rational prime-to- p Tate modules. Then

$$(3.16) \quad \widetilde{\mathcal{Z}}^\pm(m, \phi) \simeq \left[(\Gamma' \times o_k^\times) \setminus \left(\coprod_{[\mathfrak{a}] \in Cl(k)} \coprod_{\beta \in \Omega^\pm(m, \mathfrak{a}, \phi)} Z(\beta[p^\infty]) \right) \right],$$

where $\beta[p^\infty] \in \mathbb{V}_\phi^\pm$ is the quasi-morphism of p -divisible groups induced by β . Here \mathfrak{a} ranges any set of representatives of the class group $Cl(k)$ of k .

Proof. This theorem is simply a reformulation of [KR3, Proposition 6.3]; again, we content ourselves with a sketch of the argument.

First, we verify that the right hand side of (3.16) is independent of the choices of representatives $\mathfrak{a} \in Cl(k)$. Indeed, for any $a \in k^\times$, we have a bijection

$$(3.17) \quad \Omega^\pm(m, a \cdot \mathfrak{a}, \phi) \xrightarrow{\sim} \Omega^\pm(m, \mathfrak{a}, \phi), \quad \beta \mapsto p^{-\text{ord}_p n(a)/2} a \cdot \beta.$$

However, the cycles $Z(\beta[p^\infty])$ on \mathcal{D} depend only on (i) the image of $\beta[p^\infty]$ in $\mathbb{P}(\mathbb{V}_{\phi_0}^\pm)$ (which determines the horizontal part), and (ii) the p -adic valuation of $q^\pm(\beta)$, (which determines the vertical part). By construction, both of these are the same for elements β appearing on either side of (3.17).

We first consider a linear cycle $\widetilde{\mathcal{Z}}^+(m, \phi)$. Choose a set of representatives $\{\mathfrak{a}_1, \dots, \mathfrak{a}_h\}$ for $Cl(k)$, and assume, without loss of generality, that each \mathfrak{a}_i is relatively prime to (p) . Let $S \in \mathbf{Nilp}$, and let \mathfrak{a}_j denote one of our fixed representatives. Then, applying [KR3, Proposition 5.3] to the “signature (1,0) case” in the notation of loc. cit., there exists an elliptic curve $\underline{E}_j = (E_j, i_{E_j}, \lambda_{E_j}) \in \mathcal{E}^+(S)$, and an o_k -linear quasi-isogeny

$$\psi_j : E_j \rightarrow \mathbf{E}_S$$

that together are determined (uniquely up to o_k^\times) by the conditions:

- (i) $\psi_j^\vee \circ \lambda_{\mathbf{E}} \circ \psi_j = N(\mathfrak{a}_i) \cdot \lambda_{E_j}$;
- (ii) $\eta_{\mathbf{E}} \circ \psi_j(Ta^p(E_j)) = \widehat{\mathfrak{a}}_i^p$; and
- (iii) at the level of p -divisible groups over $\overline{S} = S \times_W \mathbb{F}$, we have that

$$\psi_j[p^\infty]_{\overline{S}} : E_j[p^\infty] \times_S \overline{S} \rightarrow \mathbf{E}[p^\infty] \times_{\mathbb{F}} \overline{S} = \mathbb{Y}_{\overline{S}}$$

is an isomorphism.

Furthermore, every point of $\mathcal{E}^+(S)$ is isomorphic to an elliptic curve arising in this way.

Now suppose we have a triple $(\underline{E}_j, \underline{A}, y) \in \mathcal{Z}^+(m, \phi)(S)$, for some optimal embedding ϕ , with \underline{E}_j corresponding to a fractional ideal \mathfrak{a}_j as in the previous paragraph. Define a map $\beta = \beta(y) \in \mathcal{V}_\phi^+$ by the formula:

$$\beta := \psi_A \circ y \circ \psi_j^{-1},$$

where $\psi_A : A \rightarrow \mathbf{A}$ is a quasi-isogeny chosen as in (3.10). It follows from property (i) above that

$$q^+(\beta) = \frac{h_{E_j, A}(y, y)}{N(\mathfrak{a}_j)} = \frac{m}{N(\mathfrak{a}_j)}$$

while property (ii) implies

$$\eta_{\mathbf{A}} \circ \beta^p \circ \eta_{\mathbf{E}}^{-1}(\widehat{\mathfrak{a}_j^p}) \subset \widehat{\mathcal{O}_{\mathbf{B}}^p},$$

and so $\beta \in \Omega^+(m, \mathfrak{a}, \phi)$. Finally, by construction, the point

$$(X, \iota_X, \rho_X) = \left(A[p^\infty], \iota \otimes \mathbb{Z}_p, \psi_A[p^\infty]_{\overline{S}} \right) \in \mathcal{D}(S)$$

belongs to the cycle $Z(\beta[p^\infty])$ of the previous section, c.f. Definition 2.5.

Conversely, suppose we are given an element $\beta \in \Omega^+(m, \mathfrak{a}_j, \phi)$ and a point $x \in Z(\beta[p^\infty], \phi_0)(S)$. As described in Theorem 3.10, we may associate to the point x an abelian surface $\underline{A} = (A, \iota_A)$, and an $\mathcal{O}_{\mathbf{B}}$ -linear quasi-isogeny

$$\psi_A : A \rightarrow \mathbf{A}_S.$$

Then the quasi-morphism

$$y := \psi_A^{-1} \circ \beta \circ \psi_j \in \text{Hom}(E_j, A) \otimes \mathbb{Q},$$

in fact lies in $\text{Hom}(E_j, A)$, since it induces a morphism at the level of p -divisible groups, and maps $Ta^p(E_j)$ into $Ta^p(A)$; moreover, we have $h_{E_j, A}(y, y) = m$ by construction. Hence the triple $(\underline{E}_j, \underline{A}, y)$ lies in the cycle $\tilde{\mathcal{Z}}^+(m)(S)$, and the ambiguity in the choices we have made in constructing such a tuple is catalogued by the action of the group $\Gamma' \times o_k^\times$. The claimed isomorphism (3.16) follows.

We turn now to the *anti-linear cycles* $\tilde{\mathcal{Z}}^-(m, \phi)$. Given any S -point $\underline{E} = (E, i_E, \lambda_E) \in \mathcal{E}^-(S)$, note that we obtain a point $\underline{E}' := (E, i'_E, \lambda_E) \in \mathcal{E}^+(S)$ by replacing the action i_E by its conjugate i'_E . Moreover, if we set $\phi' : o_k \rightarrow \mathcal{O}_{\mathbf{B}}$ to be the embedding conjugate to ϕ , then any o_k -linear map $y \in \text{Hom}_{o_k, \phi}(\underline{E}, \underline{A})$ also may be viewed as an o_k -linear map $y \in \text{Hom}_{o_k, \phi'}(\underline{E}', \underline{A})$; indeed, the map y remains linear after replacing the o_k -actions on both E and A by their conjugates. This observation yields an identification

$$(3.18) \quad \mathcal{Z}^-(m, \phi) \simeq \mathcal{Z}^+(m, \phi').$$

In particular, applying the uniformization we have already established above for the cycle $\tilde{\mathcal{Z}}^+(m, \phi')$ we have

$$\tilde{\mathcal{Z}}^-(m, \phi) \simeq \left[(\Gamma' \times o_k^\times) \backslash \left(\coprod_{[\mathfrak{a}] \in \text{Cl}(k)} \coprod_{\beta \in \Omega^+(m, \mathfrak{a}, \phi')} Z(\beta[p^\infty]) \right) \right].$$

From the definitions, it is evident that

$$\Omega^+(m, \mathfrak{a}, \phi') = \Omega^-(m, \mathfrak{a}, \phi);$$

we therefore obtain the uniformization (3.16) for the antilinear cycles, as required. \square

3.4. Some calculations. Our first aim is to relate the sets $\Omega^o(n)$ and $\Omega^\pm(m, \mathfrak{a}, \phi)$, in the special case that the squarefree part of n is equal to $|\Delta|$; combining these calculations with the p -adic uniformizations of the previous section allows us to prove our key result, Theorem 3.21.

We start by introducing some notation. Suppose $\xi \in \Omega^o(|\Delta|a^2)$, for some $a \in \mathbb{Z}_{>0}$. Then, by assumption, the endomorphism

$$\eta_{\mathbf{A}}^{-1} \circ \xi^p \circ \eta_{\mathbf{A}} \in \text{End}_{\mathcal{O}_B}(\widehat{\mathcal{O}_B^p})$$

is given by right-multiplication by some finite adele $(x_\ell)_{\ell \neq p} \in \widehat{\mathcal{O}_B^p}$, such that $(x_\ell)^2 = a^2 \Delta$. Therefore, for every $\ell \neq p$, we obtain an embedding

$$\varphi_\ell : k_\ell = k \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \rightarrow B_\ell, \quad a\sqrt{\Delta} \mapsto x_\ell.$$

We define the *conductor* $c = c(\xi)$ of ξ to be the smallest (rational) positive integer such that for all $\ell \neq p$, we have

$$\varphi_\ell \left(\mathbb{Z}_\ell[c\sqrt{\Delta}] \right) \subset \mathcal{O}_{B,\ell};$$

in other words, $c(\xi)$ is the smallest integer such that φ_ℓ maps the unique \mathfrak{o}_k -order of conductor $c(\xi)$ into $\mathcal{O}_{B,\ell}$, for all $\ell \neq p$. We note that since

$$\varphi_\ell \left(\mathbb{Z}_\ell[a\sqrt{\Delta}] \right) = \mathbb{Z}_\ell + \mathbb{Z}_\ell x_\ell \subset \mathcal{O}_{B,\ell},$$

we have that c divides a , and by valuation considerations it is easy to see $(c, D_B) = 1$. In particular, we obtain a disjoint decomposition

$$(3.19) \quad \Omega^o(|\Delta|a^2) = \coprod_{\substack{c|a \\ (c, D_B)=1}} \Omega^o(|\Delta|a^2, c),$$

where $\Omega^o(|\Delta|a^2, c)$ denotes the subset of elements $\xi \in \Omega^o(|\Delta|a^2)$ with $c(\xi) = c$. Note also that if $\xi \in \Omega^o(|\Delta|a^2, c)$ and t is any integer, then the conductor of $t \cdot \xi$ is again c , and one checks easily that we have a bijection

$$(3.20) \quad \Omega^o(|\Delta|a^2, c) \xrightarrow{\sim} \Omega^o(|\Delta|a^2 t^2, c), \quad \xi \mapsto t \cdot \xi.$$

Now suppose $\phi : \mathfrak{o}_k \rightarrow \mathcal{O}_B$ is an optimal embedding, and $\ell \neq p$ is a prime dividing D_B ; recall our standing assumption that such a prime is inert in k . Then, reducing modulo ℓ , we obtain two maps

$$\overline{\varphi}_\ell, \overline{\phi}_\ell : \mathfrak{o}_{k,\ell}/(\ell) \rightarrow \mathcal{O}_{B,\ell}/(\theta),$$

where $\theta \in \mathcal{O}_B$ is a fixed element such that $\theta^2 = -D_B$. As both the source and target of the maps are isomorphic to the field \mathbb{F}_{ℓ^2} , the two maps $\overline{\phi}_\ell$ and $\overline{\varphi}_\ell$ are either equal, or they differ by the Frobenius automorphism on \mathbb{F}_{ℓ^2} . This observation allows to define the *Frobenius type away from p* of ξ as follows:

$$(3.21) \quad \nu^p(\xi, \phi) := \prod_{\substack{\ell | D_B \\ \ell \neq p}} \nu_\ell(\xi, \phi), \quad \text{where} \quad \nu_\ell(\xi, \phi) := \begin{cases} 1, & \text{if } \overline{\phi}_\ell = \overline{\varphi}_\ell \\ \ell, & \text{otherwise.} \end{cases}$$

Recall that we also had the notion of a Frobenius type

$$(3.22) \quad \nu_p(\xi, \phi) := \nu_p(\xi[p^\infty], \phi_0)$$

at p , which was defined in (2.39) in terms of the parity of the p -valuation of the norm of any eigenvector of the induced map $\xi[p^\infty]$ on Dieudonné modules.

For an optimal embedding ϕ , and an element $\beta \in \mathcal{V}_\phi^+$, we may also define the notion of a conductor, as follows. Let

$$(h_\ell)_{\ell \neq p} := \eta_{\mathbf{A}} \circ \beta^p \circ \eta_{\mathbf{E}}^{-1}(1) \in B(\mathbb{A}_f^p),$$

and for each $\ell \neq p$, define an embedding $\varphi'_\ell : k_\ell \rightarrow B_\ell$ determined by the formula

$$\varphi'_\ell(\sqrt{\Delta}) = (h_\ell)^{-1} \cdot \phi(\sqrt{\Delta}) \cdot h_\ell.$$

As before, we define the *conductor* $c(\beta)$ of β to be the smallest integer c such that $\varphi'_\ell(o_c) \subset \mathcal{O}_{B,\ell}$ for all $\ell \neq p$, where $o_c = \mathbb{Z}[c\sqrt{\Delta}]$ is the unique order of conductor c . Note that this quantity depends on the choice of embedding ϕ .

Lemma 3.13. *Suppose $\beta \in \mathcal{V}_\phi^+$, and let $h = (h_\ell)_{\ell \neq p} := \eta_{\mathbf{A}} \circ \beta^p \circ \eta_{\mathbf{E}}^{-1}(1) \in B(\mathbb{A}_f^p)$. Then*

- (i) $q^+(\beta) = \Delta \operatorname{Nrd}_{B(\mathbb{A}_f^p)}(h) \in \mathbb{A}_f^p$.
- (ii) *If $\beta \in \Omega(m, \mathfrak{a}, \phi)$, then the conductor $c(\beta)$ divides m .*

Proof. (i) Recalling our fixed trivialization $\mathbb{A}_f^p(1) \simeq \mathbb{A}_f^p$, consider the diagram

$$\begin{array}{ccccc} Ta^p(\mathbf{E})_{\mathbb{Q}} \times Ta^p(\mathbf{E})_{\mathbb{Q}} & \xrightarrow{\beta^p \times \beta^p} & Ta^p(\mathbf{A})_{\mathbb{Q}} \times Ta^p(\mathbf{A})_{\mathbb{Q}} & \xrightarrow{e_{\mathbf{A}}} & \mathbb{A}_f^p(1) \simeq \mathbb{A}_f^p \\ \eta_{\mathbf{E}} \times \eta_{\mathbf{E}} \downarrow & & \eta_{\mathbf{A}} \times \eta_{\mathbf{A}} \downarrow & & \parallel \\ \mathbb{A}_{k,f}^p \times \mathbb{A}_{k,f}^p & \xrightarrow{(x,y) \mapsto (\phi(x)h, \phi(y)h)} & B(\mathbb{A}_f^p) \times B(\mathbb{A}_f^p) & \xrightarrow{(x,y) \mapsto \operatorname{Trd}(xy^t \phi(\sqrt{\Delta}))} & \mathbb{A}_f^p \end{array}$$

where $e_{\mathbf{A}}$ is the Weil pairing on \mathbf{A} defined by the polarization $\lambda_{\mathbf{A},\phi}$. It is an immediate consequence of the definitions that both squares commute. Let

$$e_{\mathbf{E}} : Ta^p(\mathbf{E})_{\mathbb{Q}} \times Ta^p(\mathbf{E})_{\mathbb{Q}} \rightarrow \mathbb{A}_f^p$$

denote the Weil pairing on \mathbf{E} defined by $\lambda_{\mathbf{E}}$. Then on the one hand, by taking adjoints, we have

$$e_{\mathbf{A}}(\beta^p(x), \beta^p(y)) = q^+(\beta) \cdot e_{\mathbf{E}}(x, y), \quad x, y \in Ta^p(\mathbf{E})_{\mathbb{Q}},$$

while on the other hand, if we write $s = \eta_{\mathbf{E}}(x)$ and $t = \eta_{\mathbf{E}}(y)$, then the commutative diagram above tells us that

$$\begin{aligned} e_{\mathbf{A}}(\beta^p(x), \beta^p(y)) &= \operatorname{Trd} \left[\phi(s)h \cdot (\phi(t)h)^t \cdot \phi(\sqrt{\Delta}) \right] \\ &= \operatorname{Nrd}(h) \cdot \operatorname{Trd} \left(\phi(s \cdot t' \cdot \sqrt{\Delta}) \right) \\ &= \operatorname{Nrd}(h) \Delta e_{\mathbf{E}}(x, y), \end{aligned}$$

where the last line follows by the choice of $\eta_{\mathbf{E}}$, as in (3.12). This proves (i).

(ii) Suppose $\beta \in \Omega^+(m, \mathfrak{a}, \phi)$. It suffices to check that

$$m \cdot (h_\ell)^{-1} \phi(\sqrt{\Delta}) h_\ell \stackrel{?}{\in} \mathcal{O}_{B,\ell}, \quad \text{for all } \ell \neq p.$$

Choose $a_\ell \in k_\ell^\times$ such that $\mathfrak{a} \otimes \mathbb{Z}_\ell = a_\ell \cdot o_{k,\ell}$. Then by the definition of $\Omega^+(m, \mathfrak{a}, \phi)$, we have $h_\ell \in \phi(a_\ell^{-1}) \cdot \mathcal{O}_{B,\ell}$, and in particular,

$$(h_\ell)^t \phi(\sqrt{\Delta}) h_\ell \in \frac{1}{N(\mathfrak{a})} \mathcal{O}_{B,\ell}.$$

Hence, by combining part (i) of the lemma with the assumption

$$q^+(\beta) = m \cdot \frac{p^{\text{ord}_p N(\mathfrak{a})}}{N(\mathfrak{a})},$$

we have

$$m \cdot (h_\ell)^{-1} \phi(\sqrt{\Delta}) h_\ell = p^{-\text{ord}_p(N(\mathfrak{a}))} \cdot \Delta \cdot N(\mathfrak{a}) \cdot \left[(h_\ell)^\iota \phi(\sqrt{\Delta}) h_\ell \right] \in \mathcal{O}_{B,\ell}$$

as required. \square

Lemma 3.14. *Let $\beta \in \Omega^+(m, \mathfrak{a}, \phi)$. Then there is a unique element $\xi = \xi(\beta) \in \Omega^o(|\Delta|m^2)$ such that*

$$\xi \circ \beta = m\sqrt{\Delta} \cdot \beta,$$

and such that the conductor $c(\xi)$ of ξ is equal to $c(\beta)$.

Proof. Note that each $\xi \in B'(\mathbb{Q})$ defines a k -linear endomorphism $[\xi]$ of \mathcal{V}_ϕ^+ by composition:

$$[\xi] \cdot \beta := \xi \circ \beta.$$

Moreover, it follows immediately from definitions that

$$q^+([\xi] \cdot \beta, [\xi] \cdot \beta) = \text{Nrd}'(\xi) \cdot q^+(\beta, \beta) \quad \text{for all } \beta \in \mathcal{V}_\phi^+.$$

Hence, if $\beta \in \Omega^+(m, \mathfrak{a}, \phi)$ and ξ_1, ξ_2 both satisfy $[\xi_i] \cdot \beta = m\sqrt{\Delta} \cdot \beta$, then $\text{Nrd}'(\xi_1 - \xi_2) = 0$, so $\xi_1 = \xi_2$, as B' is division. This proves the uniqueness claim in the lemma.

To show existence, we choose an element $\vartheta \in \mathcal{O}_B$ with $\vartheta^\iota = -\vartheta$, and such that for all $a \in k$, we have $\vartheta\phi(a) = \phi(a')\vartheta$. Then we may define a quasi-isogeny

$$\psi_\beta : \mathbf{E} \times \mathbf{E} \rightarrow \mathbf{A}, \quad (x, y) \mapsto \beta(x) + \iota_{\mathbf{A}}(\vartheta) \cdot \beta(y).$$

Note that for all $a \in o_k$, we have

$$(3.23) \quad \iota_{\mathbf{A}}(\phi(a)) \cdot \psi_\beta(x, y) = \psi_\beta(i_{\mathbf{E}}(a) \cdot x, i_{\mathbf{E}}(a') \cdot y)$$

and

$$(3.24) \quad \iota_{\mathbf{A}}(\vartheta) \cdot \psi_\beta(x, y) = \psi_\beta(\vartheta^2 y, x).$$

We may then define an element $\xi \in \text{End}(\mathbf{A})_{\mathbb{Q}}$ by the formula

$$\xi := m \cdot \psi_\beta \circ (i_{\mathbf{E}}(\sqrt{\Delta}), i_{\mathbf{E}}(\sqrt{\Delta})) \circ \psi_\beta^{-1}.$$

It follows from (3.23) and (3.24) that ξ commutes with $\iota_{\mathbf{A}}(\vartheta)$, and $\iota_{\mathbf{A}}(\phi(a))$ for all $a \in o_k$, so in fact $\xi \in B'(\mathbb{Q}) = \text{End}_{\mathcal{O}_B}(\mathbf{A})_{\mathbb{Q}}$, and it is easily seen that

$$(3.25) \quad \xi \circ \beta = m \cdot \beta \circ i_{\mathbf{E}}(\sqrt{\Delta}) = m\sqrt{\Delta} \cdot \beta.$$

It remains to verify that ξ satisfies the conditions (3.11) defining $\Omega^o(m^2|\Delta|)$. Note that by (3.25), the element ξ is not a scalar; on the other hand, $\xi^2 = m^2\Delta$ is a scalar, so we must have $\text{Trd}'(\xi) = 0$.

Finally, note that the endomorphism

$$\eta_{\mathbf{A}}^{-1} \circ \xi^p \circ \eta_{\mathbf{A}} \in \text{End}_{\mathcal{O}_B}(B(\mathbb{A}_f^p))$$

is given by right-multiplication by the element

$$x := \eta_{\mathbf{A}}^{-1} \circ \xi^p \circ \eta_{\mathbf{A}}(1) \in B(\mathbb{A}_f^p).$$

If we set

$$h := \eta_{\mathbf{A}} \circ \beta^p \circ \eta_{\mathbf{E}}^{-1}(1) \in B(\mathbb{A}_f^p),$$

then a short computation yields the implication

$$\xi \circ \beta = m \cdot \beta \circ i_{\mathbf{E}}(\sqrt{\Delta}) \implies x = m \cdot h^{-1} \phi(\sqrt{\Delta}) h.$$

By Lemma 3.13 (ii), we have $x \in \widehat{\mathcal{O}_{\mathbf{B}}^p}$, and so

$$\eta_{\mathbf{A}}^{-1} \circ \xi^p \circ \eta_{\mathbf{A}} \in \text{End}_{\mathcal{O}_{\mathbf{B}}}(\widehat{\mathcal{O}_{\mathbf{B}}^p}),$$

as required; the claim regarding conductors also follows immediately from the definitions, as does the Γ' -invariance. \square

The following lemma is a straightforward exercise in linear algebra:

Lemma 3.15. *Let $g \in M_2(\mathbb{Z}_{\ell})$ such that $\text{Tr}(g) = 0$ and $g^2 = N$, where $\text{ord}_{\ell}(N) \leq 1$. If $\ell = 2$, assume $\text{ord}_2(N) = 1$. Then there exists $u \in GL_2(\mathbb{Z}_{\ell})$ such that*

$$u \cdot g \cdot u^{-1} = \begin{pmatrix} & 1 \\ N & \end{pmatrix}.$$

\square

The next proposition gives a formula for the number of β 's that are eigenvectors for a given ξ :

Proposition 3.16. *Fix a set of representatives $\{\mathbf{a}_1, \dots, \mathbf{a}_h\}$ for the class group $Cl(k)$ of k , such that each \mathbf{a}_i is relatively prime to (p) . Consider the map*

$$f: \prod_{i=1}^h \Omega^+(m, \mathbf{a}_i, \phi) \rightarrow \Omega^o(|\Delta|m^2), \quad \beta \mapsto \xi(\beta)$$

where $\xi(\beta)$ is the unique element which has β as an eigenvector with eigenvalue $m\sqrt{\Delta}$, as in Lemma 3.14. For any $\xi \in \Omega^o(|\Delta|m^2)$, let $\nu^p = \nu^p(\xi, \phi)$ (resp. $\nu_p = \nu_p(\xi[p^\infty], \phi)$) denote its Frobenius type away from (resp. at) p , and $c = c(\xi)$ its conductor. Then we have

$$\#(f^{-1}(\xi)) = |o_k^\times| \cdot \rho\left(\frac{m}{c|\Delta|\nu_p\nu^p}\right),$$

where for any rational number N , $\rho(N)$ denotes the number of integral ideals of k of norm N .

In particular, if $m/c|\Delta|\nu_p\nu^p$ is not an integer, then the fibre $f^{-1}(\xi)$ is empty.

Proof. Let $\xi \in \Omega^o(|\Delta|m^2)$. Then, viewing ξ as a k -linear endomorphism of \mathcal{V}_ϕ^+ , it has two distinct eigenvalues $\pm m\sqrt{\Delta}$ and so there certainly exists some $\beta_0 \in \mathcal{V}_\phi^+$ such that

$$\xi \cdot \beta_0 = m\sqrt{\Delta} \cdot \beta_0.$$

Moreover, any other eigenvector with the same eigenvalue differs from β_0 by a scalar in k^\times . Hence, we have

$$\begin{aligned}
\#f^{-1}(\xi) &= \# \prod_{i=1}^h \{ \beta \in \Omega^+(m, \mathfrak{a}_i, \phi) \mid \xi(\beta) = \xi \} \\
&= \# \prod_{i=1}^h \left\{ \beta \in \mathcal{V}_\phi^+ \mid q^+(\beta) = \frac{m}{N(\mathfrak{a}_i)}, \quad \eta_{\mathbf{A}} \circ \beta^p \circ \eta_{\mathbf{E}}^{-1}(\widehat{\mathfrak{a}}_i^p) \subset \widehat{\mathcal{O}}_{\mathbf{B}}^p, \quad \xi(\beta) = \xi \right\} \\
&= \# \prod_{i=1}^h \left\{ a \in k^\times \mid q^+(a\beta_0) = \frac{m}{N(\mathfrak{a}_i)}, \quad \eta_{\mathbf{A}} \circ \beta_0^p \circ \eta_{\mathbf{E}}^{-1}(a\widehat{\mathfrak{a}}_i^p) \subset \widehat{\mathcal{O}}_{\mathbf{B}}^p \right\} \\
(3.26) \quad &= |o_k^\times| \cdot \# \left\{ \mathfrak{a} \subset k \text{ a fractional ideal} \mid N(\mathfrak{a}) = \frac{m}{q^+(\beta_0)}, \quad \eta_{\mathbf{A}} \circ \beta_0^p \circ \eta_{\mathbf{E}}^{-1}(\widehat{\mathfrak{a}}^p) \subset \widehat{\mathcal{O}}_{\mathbf{B}}^p \right\}.
\end{aligned}$$

As before, set

$$h = (h_\ell)_{\ell \neq p} := \eta_{\mathbf{A}} \circ \beta_0^p \circ \eta_{\mathbf{E}}^{-1}(1), \quad \text{and} \quad x = (x_\ell)_{\ell \neq p} := \eta_{\mathbf{A}} \circ \xi^p \circ \eta_{\mathbf{A}}^{-1}(1).$$

Then the condition $\xi \cdot \beta_0 = m\sqrt{\Delta} \cdot \beta_0$ implies

$$x_\ell = m \cdot h_\ell^{-1} \phi(\sqrt{\Delta}) h_\ell, \quad \text{for all } \ell \neq p.$$

For each $\ell \neq p$, let $\varphi_\ell : k_\ell \rightarrow B_\ell$ be the embedding determined by the relation

$$(3.27) \quad \varphi_\ell(\sqrt{\Delta}) = m^{-1} x_\ell = h_\ell^{-1} \phi(\sqrt{\Delta}) h_\ell.$$

We shall translate the conditions on the right hand side of (3.26) to a collection of local ones. First, suppose that ℓ is a prime not dividing D_B , and fix an isomorphism $\mathcal{O}_{\mathbf{B}, \ell} \simeq M_2(\mathbb{Z}_\ell)$. By applying Lemma 3.15 to the case

$$g = \phi(\sqrt{\Delta}) \in M_2(\mathbb{Z}_\ell), \quad N = \Delta,$$

and recalling that by assumption $\Delta < 0$ is squarefree and even, it follows that we may assume that the isomorphism $\mathcal{O}_{\mathbf{B}, \ell} \simeq M_2(\mathbb{Z}_\ell)$ identifies

$$(3.28) \quad \phi(\sqrt{\Delta}) \quad \text{with} \quad \begin{pmatrix} & 1 \\ \Delta & \end{pmatrix}.$$

Let $c = c(\xi)$ be the conductor of ξ , and define

$$w(c)_\ell := \begin{pmatrix} c & \\ & 1 \end{pmatrix} \in \mathcal{O}_{\mathbf{B}, \ell}.$$

Then the map

$$\varphi_c : k_\ell \rightarrow B_\ell, \quad \varphi_c(\sqrt{\Delta}) = w(c)_\ell^{-1} \phi(\sqrt{\Delta}) w(c)_\ell$$

is a local embedding of conductor c ; that is, we have

$$\varphi_c(k_\ell) \cap \mathcal{O}_{\mathbf{B}, \ell} = \varphi_c \left(\mathbb{Z}_\ell[c\sqrt{\Delta}] \right).$$

By the definition of the conductor of ξ , the same is true for the embedding $\varphi_\ell : k_\ell \rightarrow B_\ell$ as in (3.27). However, by [Vig, Theoreme II.3.2], any two embeddings of conductor c necessarily differ by an inner automorphism determined by an element of $(\mathcal{O}_{\mathbf{B}, \ell})^\times$. Furthermore, for $x, y \in B_\ell^\times$, we have

$$Ad_{x^{-1}} \circ \phi = Ad_{y^{-1}} \circ \phi \quad \Longleftrightarrow \quad x = \phi(a)y \text{ for some } a \in k_\ell^\times.$$

Hence, for each ℓ not dividing D_B , we may write

$$(3.29) \quad h_\ell = \phi(a_\ell) \cdot w(c)_\ell \cdot u_\ell$$

for some $a_\ell \in k_\ell^\times$ and $u_\ell \in (\mathcal{O}_{B,\ell})^\times$. Note moreover that $h_\ell \in (\mathcal{O}_{B,\ell})^\times$ for almost all ℓ .

Now consider a prime $\ell | D_B$, $\ell \neq p$. Fix a uniformizer $\Pi_\ell \in \mathcal{O}_{B,\ell}$ such that $\Pi_\ell \phi(a) = \phi(a') \Pi_\ell$ for all $a \in k_\ell$. Recalling our assumption that ℓ is inert in k , we may write

$$(3.30) \quad h_\ell = \phi(a_\ell) \cdot (\Pi_\ell)^{\epsilon_\ell},$$

for some $a_\ell \in k_\ell^\times$ and $\epsilon_\ell \in \{0, 1\}$. It is a straightforward exercise to prove that for the reductions

$$\overline{\varphi}_\ell, \overline{\phi} \in \text{Hom} \left(\mathcal{O}_{k,\ell}/(\ell), \mathcal{O}_{B,\ell}/(\Pi_\ell) \right),$$

we have

$$\overline{\varphi}_\ell = \overline{\phi} \iff \epsilon = 0,$$

where φ_ℓ is the embedding determined by h_ℓ as in (3.27). Hence, by definition of the Frobenius type (3.21), we have

$$\epsilon_\ell = \text{ord}_\ell(\nu_\ell(\xi, \phi)).$$

At this point, we have amassed a list of elements $(a_\ell) \in (\mathbb{A}_{k,f}^p)^\times$, as found in (3.29) and (3.30). We supplement this list with an element $a_p \in k_p^\times$ as follows: note that by the definition of the Frobenius type at p (c.f. (3.22) and (2.39)), we have

$$\text{ord}_p(\nu_p(\xi, \phi)) \equiv \text{ord}_p q^+(\beta_0) \pmod{2}.$$

We then set $a_p := p^{r/2}$, where $r = \text{ord}_p q^+(\beta_0) - \text{ord}_p \nu(\xi, \phi)$, and let \mathfrak{a}_0 denote the fractional ideal defined by $(a_\ell) \in \mathbb{A}_{k,f}^\times$.

Let $\nu_\ell = \nu_\ell(\xi, \phi)$. Note that by the product formula, we have

$$\begin{aligned} q^+(\beta_0) &= \left(\prod_{\ell \neq p} |q^+(\beta_0)|_\ell^{-1} \right) |q^+(\beta_0)|_p^{-1} \\ &= \left(\prod_{\ell \neq p} |\Delta|_\ell^{-1} \cdot |Nrd(h_\ell)|_\ell^{-1} \right) |q^+(\beta_0)|_p^{-1} \quad [\text{by Lemma 3.13}(i)] \\ &= \left(\prod_{\ell \nmid D_B} |\Delta|_\ell^{-1} |n(a_\ell)|_\ell^{-1} \cdot |Nrd(w(c)_\ell)|_\ell^{-1} \right) \left(\prod_{\substack{\ell | D_B \\ \ell \neq p}} |n(a_\ell)|_\ell^{-1} \cdot \nu_\ell \right) |n(a_p)|_p^{-1} \cdot \nu_p \\ &= |\Delta| \cdot N(\mathfrak{a}_0) \left(\prod_{\ell \neq p} |Nrd(w(c)_\ell)|_\ell^{-1} \cdot \nu_\ell \right) \cdot (\nu_p) \\ &= |\Delta| \cdot N(\mathfrak{a}_0) \cdot c \cdot \nu^p(\xi, \phi) \cdot \nu_p(\xi, \phi), \end{aligned}$$

where we have used the facts: (i) $Nrd(w(c)_\ell) = c$ for all ℓ and (ii) $|\Delta|$ is relatively prime to D_B .

At long last, we return to the quantity we wish to compute. Note that for a fractional ideal \mathfrak{a} appearing in the right hand side of (3.26), we have the equivalence

$$\eta_{\mathbf{A}} \circ \beta^p \circ \eta_{\mathbf{E}}^{-1}(\widehat{\mathfrak{a}}^p) \subset \widehat{\mathcal{O}}_{\mathbf{B}}^p \iff \phi(\widehat{\mathfrak{a}}^p) \subset \widehat{\mathcal{O}}_{\mathbf{B}}^p \cdot h^{-1}.$$

This in turn is equivalent to the collection of local statements, for all $\ell \neq p$:

$$\phi(\mathfrak{a}_\ell) \subset \mathcal{O}_{B,\ell} \cdot h_\ell^{-1} = \mathcal{O}_{B,\ell} \cdot \Pi_\ell^{-\nu_\ell} \phi(a_\ell)^{-1}, \quad \text{if } \ell \nmid D_B$$

and

$$\phi(\mathfrak{a}_\ell) \subset \mathcal{O}_{B,\ell} \cdot h_\ell^{-1} = \mathcal{O}_{B,\ell} \cdot w(c)_\ell^{-1} \cdot \phi(a_\ell)^{-1}, \quad \text{if } \ell \nmid D_B$$

Hence, replacing the ideals \mathfrak{a} appearing in (3.26) by $\mathfrak{a}_0 \cdot \mathfrak{a}$, where \mathfrak{a}_0 is the fractional ideal corresponding to $(a_\ell) \in \mathbb{A}_{k,f}^\times$, we obtain

$$(3.31) \quad \#f^{-1}(\xi) = |o_k^\times| \cdot \# \left\{ \mathfrak{a} \subset k \text{ a fractional ideal} \mid N(\mathfrak{a}) = \frac{m}{c|\Delta|\nu^p\nu_p}, \right. \\ \left. \phi(\mathfrak{a}_\ell) \subset \mathcal{O}_{B,\ell} \cdot w(c)_\ell^{-1} \text{ for } \ell \nmid D_B, \text{ and } \phi(\mathfrak{a}_\ell) \subset \mathcal{O}_{B,\ell} \Pi_\ell^{-\nu_\ell} \text{ for } \ell \mid D_B \right\}.$$

To conclude the proof, we show that an ideal \mathfrak{a} appearing on the right hand side above is necessarily integral. Note that for $\ell \mid D_B$, including the case $\ell = p$, the condition on the norm of \mathfrak{a} implies $\phi(\mathfrak{a}_\ell) \subset \mathcal{O}_{B,\ell}$. If $\ell \nmid D_B$, then with respect to the isomorphism $\mathcal{O}_{B,\ell} \simeq M_2(\mathbb{Z}_\ell)$ as in (3.28), we compute

$$\mathcal{O}_{B,\ell} \cdot w(c)_\ell^{-1} = \left\{ \begin{pmatrix} c^{-1}x & y \\ c^{-1}w & z \end{pmatrix}, x, y, w, z \in \mathbb{Z}_\ell \right\}.$$

Recall that for $a + b\sqrt{\Delta} \in k_\ell$, we have

$$\phi(a + b\sqrt{\Delta}) = \begin{pmatrix} a & b \\ b\gamma & a \end{pmatrix}, \quad \text{where } \text{ord}_\ell(\gamma) = \text{ord}_\ell(\Delta).$$

Therefore,

$$\left(\phi(k_\ell) \cap \mathcal{O}_{B,\ell} w(c)_\ell^{-1} \right) \subset \phi\left(\mathbb{Z}_\ell[\sqrt{\Delta}]\right) \subset \mathcal{O}_{B,\ell}.$$

and so, for all fractional ideals \mathfrak{a} appearing in (3.31), and all finite primes ℓ , we have $\phi(\mathfrak{a}_\ell) \subset \mathcal{O}_{B,\ell}$. As $\phi : o_k \rightarrow \mathcal{O}_B$ is an *optimal* embedding, it follows that $\mathfrak{a} \subset o_k$ is integral. \square

Before stating the main result of this section, we need a few more lemmas.

Lemma 3.17. *There exists a prime q that is split in k , and an element*

$$\varpi \in \text{End}(\mathbf{E})_{\mathbb{Q}}, \quad -Nm(\varpi) = \varpi^2 = -pq,$$

such that $\varpi \circ i_{\mathbf{E}}(a) = i_{\mathbf{E}}(a') \circ \varpi$ for all $a \in o_k$.

Proof. See [Mann, p. 144]. The idea is that since $\text{End}(\mathbf{E})_{\mathbb{Q}}$ is the quaternion algebra ramified at exactly p and ∞ , the existence of such an element ϖ is equivalent to the existence of a prime q such that

$$(-pq, \Delta)_\ell = \begin{cases} -1, & \text{if } \ell \in \{\infty, p\} \\ 1, & \text{otherwise,} \end{cases}$$

where $(\cdot, \cdot)_\ell$ is the Hilbert symbol. This imposes a finite set of congruence conditions on q , for which (infinitely many) solutions exists by Dirichlet's theorem, and furthermore such a solution is necessarily split in k . \square

Lemma 3.18. *Suppose $\phi : o_k \rightarrow \mathcal{O}_B$ is optimal, and let ϕ' denote the conjugate embedding. Then ϕ and ϕ' are not \mathcal{O}_B^\times -equivalent.*

Proof. Write $\phi' = Ad_t \circ \phi$ for some $t \in B^\times$, which is always possible by Noether-Skolem. Let ℓ be a prime dividing D_B ; then there is a uniformizer Π_ℓ such that

$$\phi' = Ad_{\Pi_\ell} \circ \phi \in Hom(o_{k,\ell}, \mathcal{O}_{B,\ell}).$$

Hence $\Pi_\ell^{-1} \cdot t \in \phi(k_\ell^\times)$, and so $ord_\ell Nrd(t)$ is necessarily odd. In particular, $t \notin \mathcal{O}_B^\times$. \square

Lemma 3.19. *Let $m \in \mathbb{Z}_{>0}$ be a positive integer, $\phi_1 : o_k \rightarrow \mathcal{O}_B$ an optimal embedding, and \mathfrak{a} a fractional ideal of k . Suppose $t \in \mathcal{O}_B$ such that $Nrd(t)$ divides $\gcd(D_B, m)$. Note that t normalizes \mathcal{O}_B , and in particular*

$$\phi_2 := Ad_{t^{-1}} \circ \phi_1$$

is again an optimal embedding. Then we have a bijection

$$\Omega^\pm(m, \mathfrak{a}, \phi_1) \xrightarrow{\sim} \Omega^\pm(m/Nrd(t), \mathfrak{a}, \phi_2), \quad \beta \mapsto \iota_{\mathbf{A}}(t^{-1}) \circ \beta.$$

Proof. Suppose $\beta \in \Omega^\pm(m, \mathfrak{a}, \phi_1)$, and set $\beta' := \iota_{\mathbf{A}}(t^{-1}) \circ \beta$; we first need to verify that

$$\beta' \stackrel{?}{\in} \Omega^\pm(m/Nrd(t), \mathfrak{a}, \phi_2).$$

Indeed, it follows immediately from definitions that if $\beta \in \mathcal{V}_{\phi_1}^\pm$, then $\beta' \in \mathcal{V}_{\phi_2}^\pm$, and that

$$q_{\phi_2}^\pm(\beta') = Nrd(t)^{-1} \cdot q_{\phi_1}^\pm(\beta) = Nrd(t)^{-1} \cdot m \cdot \frac{p^{-ord_p N(\mathfrak{a})/2}}{N(\mathfrak{a})},$$

where $q_{\phi_1}^\pm$ and $q_{\phi_2}^\pm$ are the quadratic forms on $\mathcal{V}_{\phi_1}^\pm$ and $\mathcal{V}_{\phi_2}^\pm$ defined as in (3.13) and (3.14). It only remains to check that the inclusion

$$(3.32) \quad \eta_{\mathbf{A}} \circ (\beta')^p \circ \eta_{\mathbf{E}}^{-1}(\widehat{\mathfrak{a}}^p) \stackrel{?}{\subset} \widehat{\mathcal{O}}_B^p$$

holds. Write $\widehat{\mathfrak{a}}^p = (a_\ell) \cdot \widehat{\mathcal{O}}_k^p$ for some prime-to- p idele (a_ℓ) , and set

$$h = (h_\ell)_{\ell \neq p} := \eta_{\mathbf{A}} \circ (\beta)^p \circ \eta_{\mathbf{E}}^{-1}((a_\ell)) \in \widehat{\mathcal{O}}_B^p,$$

and

$$h' = (h'_\ell) := \eta_{\mathbf{A}} \circ (\beta')^p \circ \eta_{\mathbf{E}}^{-1}((a_\ell));$$

note that to prove (3.32), it suffices to show that $h' \in \widehat{\mathcal{O}}_B^p$.

Recall that we had chosen $\eta_{\mathbf{A}} : Ta^p(\mathbf{A}) \rightarrow \widehat{\mathcal{O}}_B^p$ to be an \mathcal{O}_B -linear isomorphism, where the action of \mathcal{O}_B on $\widehat{\mathcal{O}}_B^p$ is given by left-multiplication. Hence we have

$$h' = t^{-1} \cdot h \in t^{-1} \cdot \widehat{\mathcal{O}}_B^p.$$

For primes ℓ not dividing $Nrd(t)$, note that we have $t^{-1} \cdot \mathcal{O}_{B,\ell} = \mathcal{O}_{B,\ell}$. On the other hand, suppose that ℓ divides $Nrd(t)$; in particular, ℓ divides $\gcd(D_B, m)$. Then, by Lemma 3.13, we have

$$ord_\ell Nrd(h_\ell) = ord_\ell(m) \geq 1.$$

Note that as B_ℓ is division, and $ord_\ell Nrd(t) = 1$, it follows that $h'_\ell = t^{-1} \cdot h_\ell \in \mathcal{O}_{B,\ell}$ by valuation considerations.

Thus $h'_\ell \in \mathcal{O}_{B,\ell}$ for all $\ell \neq p$, as required, and so we have shown that the assignment $\beta \mapsto \iota_{\mathbf{A}}(t^{-1}) \circ \beta$ indeed defines a map $\Omega^\pm(m, \mathfrak{a}, \phi_1) \rightarrow \Omega^\pm(m/Nrd(t), \mathfrak{a}, \phi_2)$. We note that by a similar argument, there is an inverse map

$$\Omega^\pm(m/Nrd(t), \mathfrak{a}, \phi_2) \rightarrow \Omega^\pm(m, \mathfrak{a}, \phi_1), \quad \beta' \mapsto \beta := \iota_{\mathbf{A}}(t) \circ \beta',$$

which concludes the proof of the lemma. \square

Recall that we have a decomposition $\tilde{Z}(m, \phi) = \tilde{Z}^+(m, \phi) \coprod \tilde{Z}^-(m, \phi)$ of the unitary special cycles, with p -adic uniformizations

$$(3.33) \quad \tilde{Z}^\pm(m, \phi) = \left[o_k^\times \times \Gamma' \setminus \coprod_{[\mathfrak{a}]} \coprod_{\beta \in \Omega^\pm(m, \mathfrak{a}, \phi)} Z(\beta[p^\infty]) \right]$$

as in Theorem 3.12.

Lemma 3.20. *Let $m \in \mathbb{Z}_{>0}$ be a positive integer, $\phi_1 : o_k \rightarrow \mathcal{O}_B$ an optimal embedding, and \mathfrak{a} a fractional ideal of k . Suppose $t \in \mathcal{O}_B$ such that $\text{Nrd}(t)$ divides $\gcd(D_B/p, m)$, and let $\phi_2 := \text{Ad}_{t^{-1}} \circ \phi_1$. Then we have an equality of cycles on $\tilde{\mathcal{C}}_B$:*

$$\tilde{Z}^\pm(m, \phi_1) = \tilde{Z}^\pm(m/\text{Nrd}(t), \phi_2).$$

In particular, the cycle $\tilde{Z}^\pm(m, \phi)$ only depends on the equivalence class $[\phi] \in \text{Opt}/\mathcal{O}_B^\times$ of ϕ .

Proof. Let $\beta \in \Omega^\pm(m, \mathfrak{a}, \phi_1)$, set $\beta' = \iota_{\mathbf{A}}(t^{-1}) \circ \beta$, and let $\mathbf{b} = \beta[p^\infty]$ and $\mathbf{b}' = \beta'[p^\infty]$ denote the corresponding maps on the level of p -divisible groups. Recall that for a scheme $S \in \mathbf{Nilp}$, with special fibre $\bar{S} = S \times \mathbb{F}$, we had defined the S -points of the cycle $Z(\mathbf{b})$ on \mathcal{D} to be the locus of points $(X, \iota_X, \rho_X) \in \mathcal{D}(S)$ such that the map

$$\rho_X^{-1} \circ \mathbf{b} : \mathbb{Y}_{\bar{S}} \rightarrow X_{\bar{S}}$$

lifts to a map $\mathbb{Y}_S \rightarrow X$.

Similarly, the cycle $Z(\mathbf{b}')$ parametrizes tuples (X, ι_X, ρ_X) such that $\rho_X^{-1} \circ \mathbf{b}' : \mathbb{Y}_{\bar{S}} \rightarrow X_{\bar{S}}$ lifts to a map $\mathbb{Y}_S \rightarrow X$. Note that the image of t in B_p in fact lies in $(\mathcal{O}_{B,p})^\times$ by valuation considerations, and the endomorphism

$$\iota_X(t^{-1})_{\bar{S}} : X \times \bar{S} \rightarrow X \times \bar{S}$$

evidently admits a lift to S , namely $\iota_X(t^{-1}) \in \text{End}_S(X)^\times$. Combining these observations with the fact that the quasi-isogeny ρ_X is assumed to be $\mathcal{O}_{B,p}$ -linear, we have that

$$\begin{aligned} \rho_X^{-1} \circ \mathbf{b}' &= \rho_X^{-1} \circ \iota_X(t^{-1})_{\bar{S}} \circ \mathbf{b} \text{ lifts} \iff \iota_X(t^{-1})_{\bar{S}} \circ \rho_X^{-1} \circ \mathbf{b} \text{ lifts} \\ &\iff \rho_X^{-1} \circ \mathbf{b} \text{ lifts.} \end{aligned}$$

Hence, we find that as cycles on \mathcal{D} , we have

$$Z(\mathbf{b}) = Z(\mathbf{b}').$$

The lemma then follows from Lemma 3.19 and the p -adic uniformization (3.33). \square

We now state and prove our key result:

Theorem 3.21. *Let $m \in \mathbb{Z}_{>0}$. Then we have the following equalities of cycles on $\tilde{\mathcal{C}}_B$:*

- (i) *If $|\Delta|$ does not divide m , then $\tilde{Z}(m, \phi) = 0$.*
- (ii) *If $m = |\Delta|m'$, then we have*

$$(3.34) \quad \frac{1}{2h(k)} \sum_{[\phi] \in \text{Opt}/\mathcal{O}_B^\times} \tilde{Z}(m, \phi) + \tilde{Z}\left(\frac{m}{\gcd(D_B, m)}, \phi\right) = \sum_{\substack{\alpha|m' \\ (\alpha, D_B)=1}} \chi_k(\alpha) \tilde{Z}^o\left(|\Delta| \frac{(m')^2}{\alpha^2}\right)^{\text{pure}},$$

where χ_k is the quadratic character associated to k , cf. (3.21).

Proof. (i) Suppose $|\Delta|$ does not divide m . Then for all possible values of c , ν_p , and ν^p , the quantity $m/c|\Delta|\nu_p\nu^p$ is not an integer, and so by Proposition 3.16, we have

$$\Omega^+(m, \mathfrak{a}_i, \phi) = \emptyset$$

for any fractional ideal \mathfrak{a}_i and any embedding ϕ . Hence we also have

$$\Omega^-(m, \mathfrak{a}_i, \phi) = \Omega^+(m, \mathfrak{a}_i, \phi') = \emptyset,$$

where ϕ' is the conjugate embedding. By the p -adic uniformization (3.33), it follows that $\tilde{\mathcal{Z}}^\pm(m, \phi) = \emptyset$, and so as a cycle, $\tilde{\mathcal{Z}}(m, \phi) = 0$.

(ii) Suppose $|\Delta|$ divides m , and $\phi : o_k \rightarrow \mathcal{O}_B$ is an embedding with conjugate ϕ' . By Lemma 3.18, the embeddings ϕ and ϕ' are \mathcal{O}_B^\times -inequivalent. Moreover, for any n , we have $\mathcal{Z}^+(n, \phi) = \mathcal{Z}^-(n, \phi')$, cf. (3.18). Summing over classes of optimal embeddings, it follows that

$$\begin{aligned} \frac{1}{2h(k)} \sum_{[\phi] \in \text{Opt}/\mathcal{O}_B^\times} \tilde{\mathcal{Z}}(n, \phi) &= \frac{1}{h(k)} \sum_{[\phi] \in \text{Opt}/\mathcal{O}_B^\times} \tilde{\mathcal{Z}}^+(n, \phi) \\ (3.35) \qquad \qquad \qquad &= \frac{1}{h(k)} \sum_{[\phi] \in \text{Opt}/\mathcal{O}_B^\times} \tilde{\mathcal{Z}}^-(n, \phi). \end{aligned}$$

Returning to the proof of the theorem, we proceed by cases.

Case 1: $\text{ord}_p(m) > 0$: For any cycle Z on \mathcal{D} , we let $[Z]$ denote the corresponding cycle on $\widetilde{\mathcal{C}}_B = [\Gamma' \backslash \mathcal{D}]$, and for convenience, we let

$$A := \frac{1}{2h(k)} \sum_{[\phi]} \tilde{\mathcal{Z}}(m, \phi) + \tilde{\mathcal{Z}}(m/\gcd(m, d), \phi)$$

denote the cycle we wish to compute (the left hand side of (3.34)).

Write $\gcd(m, D_B) = p \cdot \mu$, and fix an element $t \in \mathcal{O}_B$ such that $Nrd(t) = \mu$. Suppose $\phi_1 : o_k \rightarrow \mathcal{O}_B$ is an optimal embedding. By Lemma 3.20, we have a equality of cycles

$$\tilde{\mathcal{Z}}^-(m/p, \phi_1) = \tilde{\mathcal{Z}}^-\left(\frac{m}{\gcd(m, D_B)}, Ad_{t^{-1}} \circ \phi_1\right),$$

and so, upon taking the sum over all optimal embeddings and using (3.35), we obtain

$$\begin{aligned} A &= \frac{1}{h(k)} \sum_{[\phi] \in \text{Opt}/\mathcal{O}_B^\times} \tilde{\mathcal{Z}}^+(m, \phi) + \tilde{\mathcal{Z}}^-\left(\frac{m}{\gcd(m, D_B)}, \phi\right) \\ &= \frac{1}{h(k)} \sum_{[\phi]} \tilde{\mathcal{Z}}^+(m, \phi) + \tilde{\mathcal{Z}}^-(m/p, Ad_t \circ \phi) \\ &= \frac{1}{h(k)} \sum_{[\phi]} \tilde{\mathcal{Z}}^+(m, \phi) + \tilde{\mathcal{Z}}^-(m/p, \phi) \end{aligned}$$

Now suppose $\beta \in \Omega^+(m, \mathfrak{a}, \phi)$ and set

$$\beta' := \beta \circ \varpi,$$

where $\varpi \in \text{End}(\mathbf{E})$ satisfies $\varpi^2 = -pq$ for some split prime q , and $\varpi \circ i_{\mathbf{E}}(a) = i_{\mathbf{E}}(a') \circ \varpi$ for all $a \in o_k$, cf. Lemma 3.17. Then it is easily verified that

$$\beta' \in \Omega^-(m/p, \mathfrak{a}' \cdot \mathfrak{q}, \phi),$$

where \mathfrak{a}' is the conjugate of \mathfrak{a} , and \mathfrak{q} is one of the ideals above q .

Furthermore, let $\xi = \xi(\beta) \in \Omega^o(m^2|\Delta|)$ denote the unique element such that β is an eigenvector with eigenvalue $m\sqrt{\Delta}$, cf. Lemma 3.17. Let $\xi[p^\infty]$, $\beta[p^\infty]$ and $\beta'[p^\infty]$ denote the corresponding maps on p -divisible groups. Then $\beta[p^\infty]$ and $\beta'[p^\infty]$ are (up to scalars in \mathbb{Z}_p^\times) precisely the special homomorphisms \mathbf{b}^+ and \mathbf{b}^- appearing in Theorem 2.17, and the conclusion of that theorem reads

$$Z^o(\xi[p^\infty])^{pure} = Z(\beta[p^\infty]) + Z(\beta'[p^\infty]),$$

as cycles on \mathcal{D} . Hence

$$\begin{aligned} A &= \frac{1}{h(k)} \sum_{[\phi]} \tilde{Z}^+(m, \phi) + \tilde{Z}^-(m/p, \phi) \\ &= \frac{1}{h(k)} \sum_{[\phi]} \sum_{[\mathfrak{a}]} \frac{1}{|o_k^\times|} \left\{ \sum_{\substack{\beta \in \Omega^+(m, \mathfrak{a}, \phi) \\ \text{mod } \Gamma'}} [Z(\beta[p^\infty])] + \sum_{\substack{\beta' \in \Omega^-(m/p, \mathfrak{a}, \phi) \\ \text{mod } \Gamma'}} [Z(\beta'[p^\infty])] \right\} \\ &= \frac{1}{h(k)} \sum_{[\phi]} \sum_{[\mathfrak{a}]} \frac{1}{|o_k^\times|} \left\{ \sum_{\substack{\beta \in \Omega^+(m, \mathfrak{a}, \phi) \\ \text{mod } \Gamma'}} [Z(\beta[p^\infty])] + \sum_{\substack{\beta' \in \Omega^-(m/p, \mathfrak{q}\mathfrak{a}', \phi) \\ \text{mod } \Gamma'}} [Z(\beta'[p^\infty])] \right\} \\ (3.36) \quad &= \frac{1}{h(k)} \sum_{[\phi]} \sum_{[\mathfrak{a}]} \frac{1}{|o_k^\times|} \left\{ \sum_{\substack{\beta \in \Omega^+(m, \mathfrak{a}, \phi) \\ \text{mod } \Gamma'}} [Z^o(\xi[p^\infty])^{pure}] \right\} \quad \text{where } \xi = \xi(\beta). \end{aligned}$$

Given integers c , ν_p , and ν^p such that $c|m$, $\nu_p \in \{1, p\}$ and $\nu^p|(D_B/p)$, we set

$$\Omega^o(|\Delta|m^2, c, \nu, \phi) := \{ \xi \in \Omega^o(|\Delta|m^2) \mid c(\xi) = c, \nu_p(\xi, \phi) = \nu_p, \nu^p(\xi, \phi) = \nu^p \};$$

that is, the set of elements of $\Omega^o(|\Delta|m^2)$ whose conductor and Frobenius types relative to ϕ are as specified, and where, for ease of notation, we have written $\nu = \nu_p \cdot \nu^p$. Note that the action of Γ' by conjugation preserves these sets. By Proposition 3.16, we may continue:

$$A = \frac{1}{h(k)} \sum_{[\phi]} \sum_{\substack{c|m \\ (c, D_B)=1}} \sum_{\nu|D_B} \rho\left(\frac{m}{|\Delta|c\nu^p\nu_p}\right) \left\{ \sum_{\substack{\xi \in \Omega^o(m^2|\Delta|, c, \nu, \phi) \\ \text{mod } \Gamma'}} [Z^o(\xi[p^\infty])^{pure}] \right\}.$$

Writing $m = |\Delta|m'$, note that only terms with c dividing m' contribute to the above sum. Moreover, recall that all primes dividing D_B are inert in k by assumption. Let ν^* be the unique integer dividing D_B such that $\text{ord}_\ell(m'/\nu^*)$ is even for all ℓ dividing D_B ; then only terms involving $\nu = \nu^*$ can contribute to the above sum, since o_k has no ideals with norm an odd power of an inert prime. Thus we have

$$A = \frac{1}{h(k)} \sum_{\substack{c|m' \\ (c, D_B)=1}} \rho\left(\frac{m'}{c\nu^*}\right) \left\{ \sum_{[\phi]} \sum_{\substack{\xi \in \Omega^o(m^2|\Delta|, c, \nu^*, \phi) \\ \text{mod } \Gamma'}} [Z^o(\xi[p^\infty])^{pure}] \right\}.$$

Note that by [Vig, Corollaire III.5.12], we have

$$\#Opt/O_B^\times = h(k) \cdot 2^{o(D_B)},$$

where $o(D_B)$ denotes the number of prime factors of D_B . Now consider the action of the normalizer $N_{B^\times}(\mathcal{O}_B)$ on the set Opt/\mathcal{O}_B^\times , acting by conjugation. For a fixed element $\xi \in \Omega^o(m^2|\Delta|)$, the various values of the Frobenius types $\nu(\xi, \phi)$, as ϕ varies in an $N(\mathcal{O}_B)$ -orbit of optimal embeddings, will cover all $2^{o(D_B)}$ possibilities. Thus, for fixed ξ , there are exactly $h(k)$ classes $[\phi]$ of optimal embeddings such that

$$\nu(\xi, \phi) = \nu^p(\xi, \phi) \cdot \nu_p(\xi, \phi) = \nu^*.$$

Hence, it follows that

$$A = \sum_{\substack{c|m' \\ (c, D_B)=1}} \rho\left(\frac{m'}{c\nu^*}\right) \sum_{\substack{\xi \in \Omega^o(m^2|\Delta|) \\ c(\xi)=c \\ \text{mod } \Gamma'}} [Z^o(\xi[p^\infty])^{pure}].$$

Next, we claim that for any integer $N > 0$, we may write

$$\rho(N) = \sum_{a|N} \chi_k(a);$$

indeed, both sides of the above formula are multiplicative, and for $N = \ell^n$ a prime power, the fomula can immediately be verified by considering the cases ℓ split, inert, and ramified separately. Moreover, as $ord_\ell(m'/c\nu^*)$ is even for all $\ell|D_B$, it can be verified that

$$\rho\left(\frac{m'}{c\nu^*}\right) = \sum_{\substack{a|(m'/c\nu^*) \\ (a, D_B)=1}} \chi_k(a) = \sum_{\substack{a|(m'/c) \\ (a, D_B)=1}} \chi_k(a).$$

Substituting, we obtain

$$\begin{aligned} A &= \sum_{\substack{c|m' \\ (c, D_B)=1}} \sum_{\substack{a|(m'/c) \\ (a, D_B)=1}} \chi_k(a) \sum_{\substack{\xi \in \Omega^o(m^2|\Delta|) \\ c(\xi)=c \\ \text{mod } \Gamma'}} [Z^o(\xi[p^\infty])^{pure}] \\ (3.37) \quad &= \sum_{\substack{a|m' \\ (a, D_B)=1}} \chi_k(a) \cdot \sum_{\substack{c|(m'/a) \\ (c, D_B)=1}} \sum_{\substack{\xi \in \Omega^o(m^2|\Delta|) \\ c(\xi)=c \\ \text{mod } \Gamma'}} [Z^o(\xi[p^\infty])^{pure}]. \end{aligned}$$

Applying (3.20), we have that for each $a|m'$ with $(a, D_B) = 1$,

$$\begin{aligned} \sum_{\substack{c|(m'/a) \\ (c, D_B)=1}} \sum_{\substack{\xi \in \Omega^o(|\Delta|m^2) \\ c(\xi)=c \\ \text{mod } \Gamma'}} [Z^o(\xi[p^\infty])^{pure}] &= \sum_{\substack{c|(m'/a) \\ (c, D_B)=1}} \sum_{\substack{\xi \in \Omega^o(|\Delta|m'^2/a^2) \\ c(\xi)=c \\ \text{mod } \Gamma'}} [Z^o(a|\Delta| \cdot \xi[p^\infty])^{pure}]. \\ &= \sum_{\substack{c|(m'/a) \\ (c, D_B)=1}} \sum_{\substack{\xi \in \Omega^o(|\Delta|m'^2/a^2) \\ c(\xi)=c \\ \text{mod } \Gamma'}} [Z^o(\xi[p^\infty])^{pure}] \quad (\text{since } a|\Delta| \in \mathbb{Z}_p^\times) \\ &= \sum_{\substack{\xi \in \Omega^o(|\Delta|m'^2/a^2) \\ \text{mod } \Gamma'}} [Z^o(\xi[p^\infty])^{pure}] \quad (\text{by (3.19)}) \\ &= \tilde{Z}^o\left(|\Delta| \frac{(m')^2}{a^2}\right)^{pure}. \end{aligned}$$

Substituting this back into (3.37) concludes the proof of the theorem in the case $\text{ord}_p(m) > 0$.

Case 2: $\text{ord}_p(m) = 0$. As the proof is along similar lines as the previous case, we shall only indicate the necessary modifications. Fix an element $t \in \mathcal{O}_B$ with $Nrd(t) = p \cdot \gcd(m, D_B)$, and such that the image of t in $\mathcal{O}_{B,p}$ is the uniformizer Π as in Section 2. Let $\varpi \in \text{End}(\mathbf{E})_{\mathbb{Q}}$ be as in Lemma 3.17, so that in particular $\varpi^2 = -pq$ for a split prime q . Then if $\beta \in \Omega^+(m, \mathfrak{a}, \phi)$, it can be easily verified that

$$\begin{aligned} \beta' &:= \iota_{\mathbf{A}}(t) \circ \beta \circ \varpi^{-1} \in \Omega^-\left(\frac{m}{\gcd(m, D_B)}, \mathfrak{q}^{-1} \cdot \mathfrak{a}', Ad_t \circ \phi\right) \\ &= \Omega^+\left(\frac{m}{\gcd(m, D_B)}, \mathfrak{q}^{-1} \cdot \mathfrak{a}', Ad_t \circ \phi'\right), \end{aligned}$$

where \mathfrak{q} is one of the prime ideals above q , and the embedding ϕ' is the conjugate of ϕ .

Let $\xi = \xi(\beta) \in \Omega^o(|\Delta|m^2)$ denote the special endomorphism corresponding to β as in Lemma 3.14. If $j = \xi[p^\infty]$ is the corresponding map on p -divisible groups, then $\beta[p^\infty]$ and $\beta'[p^\infty]$ are equal (up to scaling by \mathbb{Z}_p^\times) to the elements \mathbf{b}_1 and \mathbf{b}_2 described in Theorem 2.19; hence, we have

$$Z^o(\xi[p^\infty])^{pure} = Z(\beta[p^\infty]) + Z(\beta'[p^\infty]).$$

Therefore, we have

$$\begin{aligned} \left(\begin{array}{l} \text{left hand side} \\ \text{of (3.34)} \end{array} \right) &= \frac{1}{2h(k)} \sum_{[\phi]} \tilde{Z}(m, \phi) + \tilde{Z}(m/\gcd(m, D_B), \phi) \\ &= \frac{1}{h(k)} \sum_{[\phi]} \tilde{Z}^+(m, \phi) + \tilde{Z}^+(m/\gcd(m, D_B), \phi) \quad [\text{by (3.35)}] \\ &= \frac{1}{h(k)} \sum_{[\phi]} \sum_{[\mathfrak{a}]} \frac{1}{|o_k^\times|} \left\{ \sum_{\substack{\beta \in \Omega^+(m, \mathfrak{a}, \phi) \\ \text{mod } \Gamma'}} [Z(\beta[p^\infty])] + \sum_{\substack{\beta' \in \Omega^+(m/(m, D_B), \mathfrak{a}, \phi) \\ \text{mod } \Gamma'}} [Z(\beta'[p^\infty])] \right\} \\ &= \frac{1}{h(k)} \sum_{[\phi]} \sum_{[\mathfrak{a}]} \frac{1}{|o_k^\times|} \left\{ \sum_{\substack{\beta \in \Omega^+(m, \mathfrak{a}, \phi) \\ \text{mod } \Gamma'}} [Z(\beta[p^\infty])] \right. \\ &\quad \left. + \sum_{\substack{\beta' \in \Omega^+(m/(m, D_B), \mathfrak{q}^{-1} \mathfrak{a}', Ad_t \circ \phi') \\ \text{mod } \Gamma'}} [Z(\beta'[p^\infty])] \right\} \\ &= \frac{1}{h(k)} \sum_{[\phi]} \sum_{[\mathfrak{a}]} \frac{1}{|o_k^\times|} \left\{ \sum_{\substack{\beta \in \Omega^+(m, \mathfrak{a}, \phi) \\ \text{mod } \Gamma'}} [Z^o(\xi[p^\infty])^{pure}] \right\} \quad \text{where } \xi = \xi(\beta). \end{aligned}$$

The proof proceeds from this point exactly as in the previous case, cf. (3.36). \square

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